A Multilevel Correction Type of Adaptive Finite Element Method for Eigenvalue Problems

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September 30, 2011

Abstract

A type of adaptive finite element method for the eigenvalue problems is proposed based on the multilevel correction scheme. In this method, adaptive finite element method to solve eigenvalue problems involves solving associated boundary value problems on the adaptive partitions and small scale eigenvalue problems on the coarsest partitions. Hence the efficiency of solving eigenvalue problems can be improved to be similar to the adaptive finite element method for the associated boundary value problems. The convergence and optimal complexity is theoretically verified and numerically demonstrated.

keywords. Eigenvalue problem, multilevel correction, adaptive finite element method, convergence, optimality

AMS Subject Classification: 65N30, 65N15, 35J25

1 Introduction

The finite element method is one of the widely used discretization schemes for solving eigenvalue problems. The adaptive finite element method (AFEM) is a meaningful approach which can generate a sequence of optimal triangulations by refining those elements where the errors, as the local error estimators indicate, are relatively large. The AFEM is really an effective way to make efficient use of given computational resources. Since Babuška and Rheinboldt [5], the AFEM has been an active topic, many researchers are attracted to study the AFEM (see, e.g., [2, 5, 6, 7, 10, 18, 35, 36, 45, 49] and the references cited therein) in the last 30 years. So far, the convergence and optimality of the AFEM for boundary value problems has been obtained and

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understood well (see, e.g., [21, 22, 34, 35, 43, 9, 33, 40, 41, 14, 13, 17, 11] and the references cited therein).

Besides for the boundary value problems, the AFEM is also a very useful and efficient way for solving eigenvalue problems (see, e.g., [8, 16, 23, 27, 28, 29, 31, 44]). The AFEM for eigenvalue problems has been analyzed in some papers (see, e.g., [20, 24, 28] and the reference cited therein). Especially, [20, 27] give an elaborate analysis of the convergence and optimality for the adaptive finite element eigenvalue computation. In [24], authors also give the analysis of the convergence for the eigenvalue problems by the AFEM.

The purpose of this paper is to propose and analyze a type of AFEM to solve the eigenvalue problems based on the recent work on the multilevel correction method (see [30]) and the two-grid correction method (see [48]). In this new scheme, the cost of solving eigenvalue problems is almost the same as solving the associated boundary value problems. Here, we adopt the techniques in [20, 27, 14] to prove the convergence and optimal complexity of the new AFEM for the eigenvalue problems. Our analysis is also based on the relationship between the finite element eigenvalue approximation and the associated boundary value problem approximation (c.f. [20, 27]).

The rest of the paper is arranged as follows. In Section 2, we shall describe some basic notation and the AFEM for the second order elliptic problems. In Section 3, we introduce a type of AFEM for the second order elliptic eigenvalue problems based on the multilevel correction scheme. The convergence analysis of this type of AFEM for eigenvalue problems will be given in Section 4 and Section 5 is devoted to proving the corresponding optimal complexity. In Section 6, some numerical experiments are presented to test the theoretical analysis. Finally, some concluding remarks are given in the last section.

2 Preliminaries

In this section, we introduce some basic notation and some useful results of AFEM for the second order elliptic boundary value problem.

Let $\Omega \subset \mathcal{R}^d$ $(d \geq 1)$ denotes a polytopic bounded domain. We use the standard notation for Sobolev space $W^{s,p}(\Omega)$ and their associated norms and seminorms (see e.g, [1, 19]). We denote $H^s(\Omega) = W^{s,2}(\Omega)$ and $H_0^1(\Omega) = \{v \in H^1(\Omega) : v|_{\partial\Omega} = 0\}$, where $v|_{\partial\Omega}$ is understood in the sense of trace, $\|v\|_{s,\Omega} = \|v\|_{s,2,\Omega}$ and $\|v\|_{0,\Omega} = \|v\|_{0,2,\Omega}$. For simplicity, following Xu [46], we use the symbol $\lesssim$ in this paper. The notation $A \lesssim B$ means that $A \leq CB$ for some constant C independent of the mesh sizes. Throughout this paper, we shall use C to denote a generic positive constant, independent of the mesh sizes, which may varies at its different occurrences. We consider finite element discretization on the shape regular family of nested conforming

meshes $\{\mathcal{T}_h\}$ over Ω: there exists a constant $\gamma*$ such that

$$\frac{h_T}{\rho_T} \leq \gamma* \qquad \forall T \in \bigcup_h \mathcal{T}_h,$$

where h_T denotes the diameter of T for each $T \in \mathcal{T}_h$, and ρ_T is the diameter of the biggest ball contained in T, $h := \max\{h_T : T \in \mathcal{T}_h\}$. In this paper, we use $\mathcal{E}_h$ to denote the set of interior faces (edges or sides) of $\mathcal{T}_h$.

The following lemma is a result of Sobolev trace theorem (e.g. [1, 37]).

Lemma 2.1. *If $s > 1/2$, then*

$$\|v\|_{0,\partial T} \lesssim h_T^{-1/2}\|v\|_{0,T} + h_T^{s-1/2}\|v\|_{s,T} \quad \forall v \in H^s(T), \ T \in \mathcal{T}_h. \tag{2.1}$$

For $f \in L^2(\Omega)$, we define the **data oscillation** (see, e.g., [33, 35]) by

$$\mathrm{osc}(f, \mathcal{T}_h) \quad := \quad \|h(f - f_h)\|_{0,\Omega} = \left(\sum_{T \in \mathcal{T}_h} \|h_T(f - \bar{f}_T)\|_{0,T}^2\right)^{1/2}, \tag{2.2}$$

where f_h denotes a piecewise polynomial approximation of f over $\mathcal{T}_h$ and $\bar{f}_T = f_h|_T$. We will denote $\bar{f}_T$ be the L^2 projection of f onto polynomials of some degree, which leads to the following inequality (see [20]):

$$\mathrm{osc}(f_1 + f_2, \mathcal{T}_h) \leq \mathrm{osc}(f_1, \mathcal{T}_h) + \mathrm{osc}(f_2, \mathcal{T}_h) \quad \forall f_1, f_2 \in L^2(\Omega). \tag{2.3}$$

2.1 Short survey on linear elliptic problem with adaptive method

In this subsection, we shall present some basic results of AFEM for the second order elliptic boundary value problem. Here, for simplicity, we consider the homogeneous boundary value problem:

$$\begin{cases} Lu := -\nabla(A \cdot \nabla u) + \varphi u &= f \quad \text{in } \Omega, \\ u &= 0 \quad \text{on } \partial\Omega, \end{cases} \tag{2.4}$$

where $A = (a_{ij})_{d \times d}$ is a symmetric positive definite matrix with $a_{ij} \in W^{1,\infty}(\Omega)$ $(i, j = 1, 2 \cdots d)$, and $0 \leq \varphi \in L^\infty(\Omega)$.

The weak form of (2.4) is: Find $u \in H_0^1(\Omega)$ such that

$$a(u, v) = (f, v) \quad \forall v \in H_0^1(\Omega), \tag{2.5}$$

where the bounded bilinear form is defined by

$$a(u, v) = \int_\Omega \left(\nabla u \cdot \nabla v + \varphi u v\right) d\Omega. \tag{2.6}$$

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From the properties of A and φ, the bilinear form $a(\cdot, \cdot)$ is bounded over $H_0^1(\Omega)$

$$|a(w, v)| \leq C_a \|w\|_{a,\Omega} \|v\|_{a,\Omega} \quad \forall w, v \in H_0^1(\Omega),$$

and satisfies

$$c_a \|w\|_{1,\Omega} \leq \|w\|_{a,\Omega} \leq C_a \|w\|_{1,\Omega},$$

where the energy norm $\|\cdot\|_{a,\Omega}$ is defined by $\|w\|_{a,\Omega} = \sqrt{a(w,w)}$, c_a and C_a are positive constants. From these properties, it is well known that (2.5) has a unique solution $u \in H_0^1(\Omega)$ for any $f \in H^{-1}(\Omega)$.

Let $V_h \subset H_0^1(\Omega)$ be the corresponding family of nested finite element spaces of continuous piecewise polynomials over $\mathcal{T}_h$ of fixed degree $m \geq 1$, which vanish on the boundary of Ω, and are equipped with the same norm $\|\cdot\|_{a,\Omega}$ of space $H_0^1(\Omega)$.

Based on the finite element space V_h, we define the finite element scheme for (2.4): Find $u_h \in V_h$ such that

$$a(u_h, v_h) = (f, v_h) \quad \forall v_h \in V_h. \tag{2.7}$$

Define the Galerkin projection $R_h : H_0^1(\Omega) \to V_h$ by

$$a(u - R_h u, v_h) = 0 \quad \forall v_h \in V_h, \tag{2.8}$$

then we have $u_h = R_h u$ and

$$\|R_h u\|_{a,\Omega} \leq \|u\|_{a,\Omega} \quad \forall u \in H_0^1(\Omega). \tag{2.9}$$

From (2.9), it is easy to get the global a priori error estimates for the finite element approximation based on the approximate properties of the finite element space V_h (c.f. [19, 48]).

In order to simplify the notation, we introduce the quantity $\eta_a(h)$ as follows:

$$\eta_a(h) = \sup_{f \in L^2(\Omega), \|f\|_{0,\Omega}=1} \inf_{v_h \in V_h} \|L^{-1}f - v_h\|_{a,\Omega}.$$

From [3, 19, 37], it is known that $\eta_a(h) \to 0$ as $h \to 0$ and the following propositions hold.

Proposition 2.1.

$$\|(I - R_h)L^{-1}f\|_{a,\Omega} \lesssim \eta_a(h)\|f\|_{0,\Omega} \quad \forall f \in L^2(\Omega) \tag{2.10}$$

and

$$\|u - R_h u\|_{0,\Omega} \lesssim \eta_a(h)\|u - R_h u\|_{a,\Omega} \quad \forall u \in H_0^1(\Omega). \tag{2.11}$$

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Next we follow the classic routine to define the a posteriori error estimator for finite element problem (2.7). Let us define the element residual $\tilde{\mathcal{R}}_T(u_h)$ and the jump residual $\tilde{\mathcal{J}}_E(u_h)$ by

$$\tilde{\mathcal{R}}_T(u_h) := f - Lu_h = f + \nabla \cdot (A\nabla u_h) - \varphi u_h \quad \text{in } T \in \mathcal{T}_h, \qquad (2.12)$$
$$\tilde{\mathcal{J}}_E(u_h) := -A\nabla u_h^+ \cdot \nu^+ - A\nabla u_h^- \cdot \nu^- := [[A\nabla u_h]]_E \cdot \nu_E \quad \text{on } E \in \mathcal{E}_h, \qquad (2.13)$$

where E is the common side of elements T^+ and T^- with outward normals ν^+ and ν^-, $\nu_E = \nu^-$, and $\omega_E := T^+ \cap T^-$ that share the same side E.

For the element $T \in \mathcal{T}_h$, we define the local error indicator $\tilde{\eta}_h(u_h, T)$ by

$$\tilde{\eta}_h(u_h, T) := \left(h_T^2 \|\tilde{\mathcal{R}}_T(u_h)\|_{0,T}^2 + \sum_{E \in \mathcal{E}_h, E \subset \partial T} h_E \|\tilde{\mathcal{J}}_E(u_h)\|_{0,E}^2 \right)^{1/2}, \qquad (2.14)$$

and the error indicator for a subdomain $\omega \subset \Omega$ by

$$\tilde{\eta}_h(u_h, \omega) := \left(\sum_{T \in \mathcal{T}_h, T \subset w} \tilde{\eta}_h^2(u_h, T) \right)^{1/2}. \qquad (2.15)$$

Thus $\tilde{\eta}_h(u_h, \Omega)$ denotes the error estimator of Ω with respect to $\mathcal{T}_h$.

Now we summarize the reliability and efficiency of the a posterior error estimator (see, e.g., [33, 35, 44]):

Lemma 2.2. ([20]) *When h_0 is small enough, there exist mesh independent constants $\tilde{C}_1, \hat{C}_2, \tilde{C}_3$ such that*

$$\|u - u_h\|_{a,\Omega} \le \tilde{C}_1 \tilde{\eta}_h(u_h, \Omega), \qquad (2.16)$$

and , for any $T \in \mathcal{T}_h$

$$\hat{C}_2^2 \tilde{\eta}_h^2(u_h, T) - \tilde{C}_3^2 \sum_{T \in \mathcal{T}_h, T \subset \omega_T} h_T^2 \|\tilde{\mathcal{R}}_T(u_h) - \overline{\tilde{\mathcal{R}}_T(u_h)}\|_{0,T}^2 \le \|u - u_h\|_{a,\omega_T}^2, \qquad (2.17)$$

where ω_T contains all the elements that share at least a side with T, $\tilde{C}_1, \hat{C}_2, \tilde{C}_3 > 0$ depends only on the shape regularity γ^*, C_a and c_a, $\bar{\omega}$ is the L^2-projection of w onto polynomials of degree m on T.

As a consequence of (2.17), we have

$$\tilde{C}_2^2 \tilde{\eta}_h^2(u_h, \Omega) - \tilde{C}_3^2 \text{osc}(f - L(u_h), \mathcal{T}_h)^2 \le \|u - u_h\|_{a,\Omega}^2, \qquad (2.18)$$

where $\tilde{C}_2 = \frac{\hat{C}_2}{\check{C}_0}$ and $\check{C}_0$ is a constant depending on the shape regularity of the mesh $\mathcal{T}_h$.

It is obvious that if the right hand side term f of (2.4) is a piecewise polynomial function over $\mathcal{T}_h$, (2.18) can be simplified to

$$\tilde{C}_2^2 \tilde{\eta}_h^2(u_h, \Omega) - \tilde{C}_3^2 \text{osc}(L(u_h), \mathcal{T}_h)^2 \leq \|u - u_h\|_{a,\Omega}^2. \tag{2.19}$$

There are adaptive algorithms in [14, 21, 33, 35] to solve (2.7), which introduce two type of marking strategies to promise reduction of both error and oscillation. For these two type of methods, both convergence and optimal complexity of the adaptive finite element algorithm have been obtained (see, e.g., [21, 33, 35, 40, 41]). However, the oscillation marking is not necessary which has been proved by Cascon et al. [14]. Thus, the adaptive algorithm without oscillation marking which is adopted in this paper can be stated as follows (c.f. [14, 20]).

Adaptive Algorithm C_0

Choose parameter $0 < \theta < 1$:
1. Let $k = 0$, pick an initial mesh $\mathcal{T}_{h_0}$ and start the loop.
2. On the mesh $\mathcal{T}_{h_k}$, solve the problem (2.7) for the discrete solution u_{h_k}.
3. Compute the local indicators $\tilde{\eta}_{h_k}(u_h, T)$.
4. Construct the submesh $\widehat{\mathcal{T}}_{h_k} \subset \mathcal{T}_{h_k}$ by **Marking Strategy** E_0 with parameters θ.
5. Refine $\mathcal{T}_{h_k}$ to generate a new conforming mesh $\mathcal{T}_{h_{k+1}}$ by procedure **REFINE**.
6. Let $k = k + 1$ and go to step 2.

As in [14], the procedure **REFINE** used in **Adaptive Algorithm** C_0 is not required to satisfy the Interior Node Property of [33, 35]. Here we use the iterative or recursive bisection (see, e.g., [32, 42]) of elements with the minimal refinement condition in the procedure **REFINE**. The **Marking Strategy** adopted in **Adaptive Algorithm** C_0 was introduced by Dörfler [21] and Morin et al. [35] and can be defined as follows.

Marking Strategy E_0

Given parameter $0 < \theta < 1$:
1. Construct a minimal subset $\widehat{\mathcal{T}}_H$ from $\mathcal{T}_H$ by selecting some elements in $\mathcal{T}_H$ such that

$$\sum_{T \in \widehat{\mathcal{T}}_H} \tilde{\eta}_H^2(u_h, T) \geq \theta \tilde{\eta}_H^2(u_h, \Omega).$$

2. Mark all the elements in $\widehat{\mathcal{T}}_H$.

In order to analyze the convergence of the AFEM, we need the following lemma.

Lemma 2.3. *([20]) There exits a constant C_* only depending on the equation parameters and the mesh regularity γ^* such that*

$$\operatorname{osc}(L(v_h), \mathcal{T}_h) \leq C_L \|v_h\|_{a,\Omega} \quad \forall v_h \in V_h. \tag{2.20}$$

The convergence of **Adaptive Algorithm** C_0 has been proved by Cascon et al [14] and can be stated as follows.

Theorem 2.1. *([14]) Let $\{u_{h_k}\}_{k \in \mathbb{N}_0}$ be a sequence finite element solutions of (2.4) based on the sequence of nested meshes $\{\mathcal{T}_{h_k}\}_{k \in \mathbb{N}_0}$ produced by **Adaptive Algorithm** C_0. Then, there exist constants $\tilde{\gamma}$ and $\xi \in (0,1)$, depending on the shape regularity of meshes, the data and the parameters used in **Adaptive Algorithm** C_0, such that any two consecutive iterates k and $k+1$ have the property*

$$\|u - u_{h_{k+1}}\|_{a,\Omega}^2 + \tilde{\gamma}\tilde{\eta}_{h_{k+1}}^2(u_{h_{k+1}}, \Omega) \leq \xi^2 \Big(\|u - u_{h_k}\|_{a,\Omega}^2 + \tilde{\gamma}\tilde{\eta}_{h_k}^2(u_{h_k}, \Omega) \Big), \tag{2.21}$$

where $\mathbb{N}_0 = \{0, 1, 2, 3, \cdots\}$ and the constant $\tilde{\gamma}$ has the following form

$$\tilde{\gamma} = \frac{1}{(1 + \delta^{-1})C_L^2}, \tag{2.22}$$

with some constant $\delta \in (0, 1)$.

3 The eigenvalue problem and adaptive finite element method based on multilevel correction

In this section, we introduce a type of AFEM based on multilevel correction scheme for the linear second order elliptic eigenvalue problem.

We are concerned with the following eigenvalue problem

$$\begin{cases} -\nabla(A \cdot \nabla u) + \varphi u &= \lambda u & \text{in } \Omega, \\ u &= 0 & \text{on } \partial\Omega, \\ \int_\Omega u^2 d\Omega &= 1. \end{cases} \tag{3.1}$$

The corresponding weak form can be written as: Find $(\lambda, u) \in \mathcal{R} \times H_0^1(\Omega)$ such that $\|u\|_{0,\Omega} = 1$ and

$$a(u, v) = \lambda(u, v) \quad \forall v \in H_0^1(\Omega). \tag{3.2}$$

As we know the eigenvalue problem (3.2) has a countable sequence of real eigenvalues

$$0 < \lambda_1 < \lambda_2 \leq \lambda_3 \leq \cdots$$

and corresponding orthogonal eigenfunctions

$$u_1, u_2, u_3, \cdots,$$

which satisfy $(u_i, u_j) = \delta_{ij}, i, j = 1, 2, \cdots$.

Now we state an useful Rayleigh quotient expansion of the eigenvalue which is expressed by the eigenfunction approximation (see [3, 30, 48]).

Lemma 3.1. *Let (λ, u) be an eigenpair of (3.2). Then for any $w \in H_0^1(\Omega)$, we have*

$$\frac{a(w, w)}{(w, w)} - \lambda = \frac{a(w - u, w - u)}{(w, w)} - \frac{\lambda(w - u, w - u)}{(w, w)}. \tag{3.3}$$

The standard finite element discretization for (3.2) is: Find $(\lambda_h, u_h) \in \mathcal{R} \times V_h$ such that $\|u_h\|_0 = 1$ and

$$a(u_h, v_h) = \lambda_h(u_h, v_h) \quad \forall v_h \in V_h. \tag{3.4}$$

We can also order the eigenvalues of (3.4) as an increasing sequence

$$0 < \lambda_{1,h} < \lambda_{2,h} \le \lambda_{3,h} \le \cdots \le \lambda_{n_h,h}, \quad n_h = \dim V_h,$$

and the corresponding orthogonal eigenfunctions

$$u_{1,h}, \ u_{2,h}, \ u_{3,h}, \ \cdots, \ u_{n_h,h}$$

satisfying $(u_{i,h}, u_{j,h}) = \delta_{ij}, i, j = 1, 2, \cdots, n_h$.

From the minimum-maximum principle (see [3, 15]) and Lemma 3.1, we have

$$\lambda_i \le \lambda_{i,h} \le \lambda_i + \bar{C}_i \|u_i - u_{i,h}\|_{a,\Omega}^2, \quad i = 1, 2, , \cdots, n_h \tag{3.5}$$

with $\bar{C}_i$ constants independent of mesh size h.

3.1 Adaptive multilevel correction algorithm for eigenvalue problem

The adaptive procedure consists of loops of the form

$$\textbf{Solve} \rightarrow \textbf{Estimate} \rightarrow \textbf{Mark} \rightarrow \textbf{Refine}.$$

Similarly to **Marking Strategy** E_0, we define **Marking Strategy** E for (3.4) to enforce the error reduction as follows:

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Marking Strategy E

Given a parameter $0 < \theta < 1$:

1. Construct a minimal subset $\widehat{\mathcal{T}}_H$ from $\mathcal{T}_H$ by selecting some elements in $\mathcal{T}_H$ such that

$$\sum_{T \in \widehat{\mathcal{T}}_H} \eta_H^2(u_h, T) \geq \theta \eta_H^2(u_h, \Omega),$$

where $\eta_h(u_h, T)$ and $\eta_h(u_h, \Omega)$ denote the error indicator of the eigenfunction approximation u_h on T and Ω, respectively.

2. Mark all the elements in $\widehat{\mathcal{T}}_H$.

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Then we present a type of AFEM to compute the eigenvalue problem in the multilevel correction framework.

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Adaptive Algorithm C

1. Pick up an initial mesh $\mathcal{T}_{h_0}$ with mesh size h_0.

2. Construct the finite element space V_{h_0} and solve the following eigenvalue problem to get the discrete solution $(\lambda_{h_0}, u_{h_0}) \in \mathcal{R} \times V_{h_0}$ such that $\|u_{h_0}\|_{0,\Omega} = 1$ and

$$a(u_{h_0}, v_{h_0}) \;=\; \lambda_{h_0}(u_{h_0}, v_{h_0}) \quad \forall v_{h_0} \in V_{h_0}. \tag{3.6}$$

3. Let $k = 0$.

4. Compute the local error indicators $\eta_{h_k}(u_{h_k}, T)$.

5. Construct $\widehat{\mathcal{T}}_{h_k} \subset \mathcal{T}_{h_k}$ by **Marking Strategy E** and parameter θ.

6. Refine $\mathcal{T}_{h_k}$ to get a new conforming mesh $\mathcal{T}_{h_{k+1}}$ by procedure **Refine**.

7. Solve the following source problem on $\mathcal{T}_{h_{k+1}}$ for the discrete solution $\widetilde{u}_{h_{k+1}} \in V_{h_{k+1}}$:

$$a(\widetilde{u}_{h_{k+1}}, v_{h_{k+1}}) \;=\; \lambda_{h_k}(u_{h_k}, v_{h_{k+1}}) \quad \forall v_{h_k} \in V_{h_k}. \tag{3.7}$$

8. Construct the new finite element space $V_{h_0, h_{k+1}} = V_{h_0} + \text{span}\{\widetilde{u}_{h_{k+1}}\}$ and solve the eigenvalue problem to get the solution $(\lambda_{h_{k+1}}, u_{h_{k+1}}) \in \mathcal{R} \times V_{h_0, h_{k+1}}$ such that $\|u_{h_{k+1}}\|_{0,\Omega} = 1$ and

$$a(u_{h_{k+1}}, v_{h_{h_0, h_{k+1}}}) = \lambda_{h_{k+1}}(u_{h_{k+1}}, v_{h_0, h_{k+1}}) \quad \forall v_{h_0, h_{k+1}} \in V_{h_0, h_{k+1}}. \tag{3.8}$$

9. Let $k = k + 1$ and go to Step 4.

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Local error indicator $\eta_h(u_{h_k}, T)$ in **Adaptive Algorithm** C will be given in the next subsection. For the aim of error estimate, we define

$$M(\lambda_i) = \{ w \in H_0^1(\Omega) : w \text{ is an eigenfunction of (3.1)} \\ \text{corresponding to the eigenvalue } \lambda_i \}$$

and the quantity

$$\delta_h(\lambda_i) = \sup_{w \in M(\lambda_i), \|w\|_{0,\Omega} = 1} \inf_{v_h \in V_h} \|w - v_h\|_{a,\Omega}.$$

In the following analysis, we only need some crude priori error estimates stated as follows.

Lemma 3.2. *The obtained eigenpair approximation* (λ_{h_k}, u_{h_k}) $(k = 0, 1, \cdots)$ *after each adaptive step in* **Adaptive Algorithm** C *has the error estimate*

$$\|u - u_{h_k}\|_{a,\Omega} \lesssim \delta_{h_0}(\lambda), \tag{3.9}$$
$$\|u - u_{h_k}\|_{0,\Omega} \lesssim \eta_a(h_0) \|u - u_{h_k}\|_{a,\Omega}, \tag{3.10}$$
$$|\lambda - \lambda_{h_k}| \lesssim \|u - u_{h_k}\|_{a,\Omega}^2. \tag{3.11}$$

Proof. Based on the error estimate theory of eigenvalue problem by finite element method (c.f. [3, 4]), the eigenfunction approximation of problem (3.6) or (3.8) has the following estimates

$$\|u - u_{h_k}\|_{a,\Omega} \lesssim \sup_{w \in M(\lambda)} \inf_{v_h \in V_{h_0,h_k}} \|w - v_h\|_{a,\Omega} \\ \lesssim \sup_{w \in M(\lambda)} \inf_{v_h \in V_{h_0}} \|w - v_h\|_{a,\Omega} \lesssim \delta_{h_0}(\lambda). \tag{3.12}$$

and

$$\|u - u_{h_k}\|_{0,\Omega} \lesssim \widetilde{\eta}_a(h_0) \|u - u_{h_k}\|_{a,\Omega}, \tag{3.13}$$

where

$$\widetilde{\eta}_a(h_0) = \sup_{f \in V, \|f\|_{0,\Omega} = 1} \inf_{v \in V_{h_0,h_k}} \|L^{-1}f - v\|_{a,\Omega} \le \eta_a(h_0). \tag{3.14}$$

From (3.12), (3.13), and (3.14), we can obtain (3.9) and (3.10). The estimate (3.11) can be derived by Lemma 3.1 and (3.10). $\qquad \square$

3.2 A posteriori error estimate for eigenvalue problem

Now, we are going to give an a posteriori error estimator for the eigenvalue problem. The a posteriori error estimators have been studied extensively (see, e.g.,

[8, 20, 23, 27, 28, 29, 31]). Here we use the similar way in [20] to derive the a posteriori error estimator for the eigenvalue problem by **Adaptive Algorithm** C from a relationship between the elliptic eigenvalue approximation and the associated boundary value approximation.

In this paper, we set $H = h_k$ and $h = h_{k+1}$. Let $K : L^2(\Omega) \to H_0^1(\Omega)$ be the operator defined by

$$a(Kw, v) = (w, v) \qquad \forall w, v \in H_0^1(\Omega). \tag{3.15}$$

Then the eigenvalue problems (3.2) and (3.4) can be written as

$$u = \lambda K u, \quad u_h = \lambda_h R_h K u_h. \tag{3.16}$$

In the step 7 of **Adaptive Algorithm** C, we can view (3.7) as the finite element approximation of the problem

$$a(w^H, v) \;=\; (\lambda_H u_H, v) \quad \forall v \in H_0^1(\Omega). \tag{3.17}$$

Thus we have $w^H = \lambda_H K u_H$ and

$$\widetilde{u}_h = R_h w^H. \tag{3.18}$$

Similarly we can also define w^h as

$$a(w^h, v) \;=\; (\lambda_h u_h, v) \quad \forall v \in H_0^1(\Omega). \tag{3.19}$$

Let $r(h_0) := \eta_a(h_0) + \|u - u_H\|_{a,\Omega}$. Obviously from Lemma 3.2, we know $r(h_0) \ll 1$ if h_0 is small enough.

Theorem 3.1. *We have the following estimate*

$$\|u - u_h\|_{a,\Omega} = \|w^h - R_h w^h\|_{a,\Omega} + \mathcal{O}(r(h_0))(\|u - u_H\|_{a,\Omega} + \|u - u_h\|_{a,\Omega}). \tag{3.20}$$

Proof. From the definition (3.8), we have

$$
\begin{aligned}
u - u_h \;&=\; u - w^h + w^h - R_h w^h + R_h w^h - R_h w^H \\
&\quad + R_h w^H - u_h \\
&=\; u - w^h + w^h - R_h w^h + R_h(w^h - w^H) \\
&\quad + \widetilde{u}_h - u_h.
\end{aligned} \tag{3.21}
$$

The following equality holds

$$u - w^h \;=\; \lambda K u - \lambda_h K u_h = \lambda K(u - u_h) + (\lambda - \lambda_h) K u_h, \tag{3.22}$$

which together with the fact $\|K(u - u_h)\|_{a,\Omega} \lesssim \|u - u_h\|_{0,\Omega}$, (3.10), and (3.11) leads to

$$\|u - w^h\|_{a,\Omega} \;\leq\; \tilde{C} r(h_0) \|u - u_h\|_{a,\Omega}. \tag{3.23}$$

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Similarly, we have

$$
\begin{aligned}
\|R_h w^h - R_h w^H\|_{a,\Omega} &\leq \|w^h - w^H\|_{a,\Omega} = \|\lambda_h K u_h - \lambda_H K u_H\|_{a,\Omega} \\
&\leq \tilde{C} r(h_0)(\|u - u_h\|_{a,\Omega} + \|u - u_H\|_{a,\Omega}). \quad (3.24)
\end{aligned}
$$

Since $u_h - \widetilde{u}_h \in V_{h_0,h}$, (3.10), and (3.11), the following inequality holds

$$
\begin{aligned}
a(u_h - \widetilde{u}_h, u_h - \widetilde{u}_h) &= (\lambda_h u_h - \lambda_H u_H, u_h - \widetilde{u}_h) \\
&\leq |\lambda_h - \lambda_H|(u_h, u_h - \widetilde{u}_h) + \lambda_H(u_h - u_H, u_h - \widetilde{u}_h) \\
&\leq \tilde{C} r(h_0)(\|u - u_h\|_{a,\Omega} + \|u - u_H\|_{a,\Omega})\|u_h - \widetilde{u}_h\|_{a,\Omega}. \quad (3.25)
\end{aligned}
$$

Thus with the coercivity of $a(\cdot,\cdot)$, we have

$$
\|u_h - \widetilde{u}_h\|_{a,\Omega} \leq \tilde{C} r(h_0)(\|u - u_h\|_{a,\Omega} + \|u - u_H\|_{a,\Omega}). \quad (3.26)
$$

Finally from (3.21), (3.23), (3.24), and (3.26), the desired result (3.20) can be obtained and the proof is complete. $\qquad\square$

Theorem 3.1 builds a basic relationship between $\|u - u_h\|$ and $\|w^h - R_h w^h\|$, the former is the error between the ture and the discrete eigenfunctions, while the latter the error between w^h and its finite element projection, which has been well analyzed. Since the difference between $\|u - u_h\|$ and $\|w^h - R_h w^h\|$ is a higher order term, as in [20], we follow the procedure of the analysis of convergence and complexity for the source problem.

We define the element residual $\mathcal{R}_T(u_h)$ and the jump residual $\mathcal{J}_E(u_h)$ as follow:

$$
\mathcal{R}_T(u_h) := \lambda_h u_h - L(u_h) = \lambda_h u_h + \nabla \cdot (A\nabla u_h) - \varphi u_h \quad \text{in } T \in \mathcal{T}_h, \quad (3.27)
$$

$$
\mathcal{J}_E(u_h) := -A\nabla u_h^+ \cdot \nu^+ - A\nabla u_h^- \cdot \nu^- := [[A\nabla u_h]]_E \cdot \nu_E \quad \text{on } E \in \mathcal{E}_h, \quad (3.28)
$$

where E, ν^+ and ν^- are defined as those of Sect. 2.1.

For each element $T \in \mathcal{T}_h$, we define the local error indicator $\eta_h(u_h, T)$ by

$$
\eta_h^2(u_h, T) := h_T^2 \|\mathcal{R}_T(u_h)\|_{0,T}^2 + \sum_{E \in \mathcal{E}_h, E \subset \partial T} h_E \|\mathcal{J}_E(u_h)\|_{0,E}^2. \quad (3.29)
$$

Then on a subset $\omega \subset \Omega$, we define the error estimator $\eta_h(u_h, \omega)$ by

$$
\eta_h(u_h, \omega) := \left(\sum_{T \in \mathcal{T}_h, T \subset w} \eta_h^2(u_h, T) \right)^{1/2}. \quad (3.30)
$$

As same as (2.3) and (2.20), we have the similar inequalities of the indicator $\eta_h(v_h, \omega)$ for any $v_h \in V_h$.

Lemma 3.3. *The following inequalities for the indicator $\eta_h(v_h, \omega)$ hold*

$$
\eta_h(w_h + v_h, \omega) \leq \eta_h(w_h, \omega) + \eta_h(v_h, \omega) \quad \forall w_h \in V_h, v_h \in V_h, \quad (3.31)
$$

$$
\eta_h(v_h, \Omega) \leq C_R \|v_h\|_{a,\Omega} \quad \forall v_h \in V_h. \quad (3.32)
$$

Proof. The first inequality (3.31) can be obtained from the definition of η_h. Now we prove the second inequality (3.32).

It is obvious that the inverse estimate implies

$$\|L(v_h)\|_{0,T} \leq C_A h_T^{-1} \|\nabla v_h\|_{0,T} + C_c \|v_h\|_{0,T} \quad \forall T \in \mathcal{T}_h, \tag{3.33}$$

where C_A depends on A and the shape regularity constant γ^*, C_c depends on the coefficient φ. Namely, there exist some constants $\tilde{C}_T$ and C_R depending on C_A and C_c such that

$$\sum_{T \in \mathcal{T}_h} h_T^2 \|L(v_h)\|_{0,T}^2 \leq \sum_{T \in \mathcal{T}_h} \tilde{C}_T^2 \|v_h\|_{1,T}^2 \leq C_R^2 \|v_h\|_{a,\Omega}^2. \tag{3.34}$$

From the trace inequality (2.1) and the inverse estimate, we have

$$
\begin{aligned}
h_E \big\| [[A\nabla v_h]]_E \cdot \nu_E \big\|_{0,E}^2 &\leq \tilde{C}_E \|A\nabla v_h\|_{0,\omega_E}^2 + \tilde{C}_E h_{T^+}^2 \|A\nabla v_h\|_{1,T^+}^2 \\
&\qquad + \tilde{C}_E h_{T^-}^2 \|A\nabla v_h\|_{1,T^-}^2 \\
&\leq C_E^2 \|v_h\|_{1,\omega_E}^2,
\end{aligned} \tag{3.35}
$$

where $\omega_E := T^+ \cup T^-$ denotes the patch including the elements sharing the edge E and the constant C_E depends on A and the shape regularity constant γ^*.

Hence the desired result (3.32) can be obtained from (3.29), (3.34), and (3.35) and the proof is complete. $\square$

Based on Theorem 3.1 and Lemma 3.3, the error estimate $\eta_h(u_h, \Omega)$ has the following properties.

Theorem 3.2. *Let h_0 be small enough and $h \in (0, h_0]$. Then there are mesh independent constants such that*

$$\|u - u_h\|_{a,\Omega} \leq C_1 \eta_h(u_h, \Omega) + \mathcal{O}(r(h_0)) \|u - u_H\|_{a,\Omega}, \tag{3.36}$$

and

$$C_2^2 \eta_h^2(u_h, \Omega) - C_3^2 \mathrm{osc}(L(u_h), \mathcal{T}_h)^2 \leq \|u - u_h\|_{a,\Omega}^2 + \mathcal{O}(r^2(h_0)) \|u - u_H\|_{a,\Omega}^2. \tag{3.37}$$

Consequently we have

$$|\lambda - \lambda_h| \lesssim \eta_h^2(u_h, \Omega) + \mathcal{O}(r^2(h_0)) \|u - u_H\|_{a,\Omega}^2, \tag{3.38}$$

and

$$\eta_h^2(u_h, \Omega) - \mathrm{osc}(L(u_h), \mathcal{T}_h)^2 \lesssim |\lambda - \lambda_h| + \mathcal{O}(r^2(h_0)) \|u - u_H\|_{a,\Omega}^2. \tag{3.39}$$

Proof. From (2.16), (3.20), (3.21), (3.24), (3.26), (3.31), and (3.32), we have

$$
\|w^h - R_h w^h\|_{a,\Omega} \leq \tilde{C}_1 \eta_h(R_h w^h, \Omega) = \tilde{C}_1 \eta_h(u_h + R_h w^h - u_h, \Omega)
$$
$$
\leq \ \tilde{C}_1 \eta_h(u_h, \Omega) + \tilde{C}_1 \eta_h(R_h w^h - u_h, \Omega)
$$
$$
\leq \ \tilde{C}_1 \eta_h(u_h, \Omega) + \tilde{C}_1 C_R \|R_h w^h - u_h\|_{a,\Omega}
$$
$$
\leq \ \tilde{C}_1 \eta_h(u_h, \Omega) + \tilde{C}_1 C_R r(h_0)(\|u - u_h\|_{a,\Omega} + \|u - u_H\|_{a,\Omega}), \tag{3.40}
$$

which together with (3.20) leads to the desired result (3.36) with

$$
C_1 = \tilde{C}_1. \tag{3.41}
$$

Combing (2.19), (2.20), (3.26), (3.31), and (3.32) leads to

$$
\tilde{C}_2^2 \eta_h^2(u_h, \Omega) - \tilde{C}_3^2 \mathrm{osc}(L(u_h), \mathcal{T}_h)^2
$$
$$
\leq \ \tilde{C}_2^2 \tilde{\eta}_h^2(R_h w^h, \Omega) - \tilde{C}_3^2 \mathrm{osc}(L(R_h w^h), \mathcal{T}_h)^2
$$
$$
+ (C_L^2 + C_R^2)(\tilde{C}_2^2 + \tilde{C}_3^2)\|u_h - R_h w^h\|_{a,\Omega}^2
$$
$$
\leq \ \|w^h - R_h w^h\|_{a,\Omega}^2 + \tilde{C}\|u_h - R_h w^h\|_{a,\Omega}^2
$$
$$
\leq \ 2\|u - u_h\|_{a,\Omega}^2 + Cr^2(h_0)(\|u - u_h\|_{a,\Omega}^2 + \|u - u_H\|_{a,\Omega}^2), \tag{3.42}
$$

where the constant C depends on $\tilde{C}$, C_L, C_R, $\tilde{C}_2$ and $\tilde{C}_3$. From (3.42), the desired result (3.37) can obtained with

$$
C_2^2 = \frac{\tilde{C}_2^2}{2 + Cr^2(h_0)}, \quad C_3^2 = \frac{\tilde{C}_3^2}{2 + Cr^2(h_0)}. \tag{3.43}
$$

From Lemma 3.1 and (3.36), we obtain (3.38) and (3.39) can be derived from (3.3) and (3.37). □

4   Convergence of adaptive finite element method for eigenvalue problem

In this section, we give the convergence analysis of the **Adaptive Algorithm** C for the eigenvalue problem.

Before establishing the error reduction of the **Adaptive Algorithm** C for the eigenvalue problem, we give some preparations.

Similarly to Theorem 3.1, we also give some relationships between two level approximations, which will be used in the following analysis.

Lemma 4.1. *Let* $h, H \in (0, h_0]$. *If* $w^h = \lambda_h K u_h$, $w^H = \lambda_H K u_H$, *we have*

$$
\|u - u_h\|_{a,\Omega} \ = \ \|w^H - R_h w^H\|_{a,\Omega}
$$

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$$+\mathcal{O}(r(h_0))\big(\|u - u_h\|_{a,\Omega} + \|u - u_H\|_{a,\Omega}\big), \qquad (4.1)$$

$$\mathrm{osc}(L(u_h), \mathcal{T}_h) = \mathrm{osc}(L(R_h w^H), \mathcal{T}_h)$$

$$+\mathcal{O}(r(h_0))(\|u - u_H\|_{a,\Omega} + \|u - u_h\|_{a,\Omega}), \qquad (4.2)$$

and

$$\eta_h(u_h, \Omega) = \tilde{\eta}_h(R_h w^H, \Omega) + \mathcal{O}(r(h_0))(\|u - u_h\|_{a,\Omega} + \|u - u_H\|_{a,\Omega}). \quad (4.3)$$

Proof. First we have

$$u - u_h = u - w^H + w^H - R_h w^H + R_h w^H - u_h. \qquad (4.4)$$

Similarly to (3.23), the following inequality holds

$$\|u - w^H\|_{a,\Omega} = \|u - K\lambda_H u_H\|_{a,\Omega} = \|K\lambda u - K\lambda_H u_H\|_{a,\Omega}$$

$$\lesssim \|\lambda u - \lambda_H u_H\|_{0,\Omega} = \mathcal{O}\big(r(h_0)\big)\|u - u_H\|_{a,\Omega} \qquad (4.5)$$

Combining (3.26), (4.4), and (4.5) leads to (4.1).

By the property (2.20) of oscillation, we have

$$\mathrm{osc}(L(R_h w^H - u_h), \mathcal{T}_h) \lesssim C_L \|R_h w^H - u_h\|_{a,\Omega},$$

which together with (3.26) and the fact $\tilde{u}_h = R_h w^H$ implies

$$\mathrm{osc}(L(R_h w^H - u_h), \mathcal{T}_h) \lesssim r(h_0)(\|u - u_H\|_{a,\Omega} + \|u - u_h\|_{a,\Omega}). \qquad (4.6)$$

Hence from (2.20), (4.6), and

$$L(u_h) = L(R_h w^h) + L(R_h w^H - u_h), \qquad (4.7)$$

we can obtain the desired result (4.2).

Now we come to consider the relation (4.3). Using (3.26), (3.32), and the fact $\tilde{u}_h = R_h w^H$, we obtain

$$\tilde{\eta}_h(R_h w^H - u_h, \Omega) \lesssim r(h_0)(\|u - u_H\|_{a,\Omega} + \|u - u_h\|_{a,\Omega}). \qquad (4.8)$$

Combining (3.31), (4.8) and the fact

$$\eta_h(u_h, \Omega) = \eta_h(R_h w^H + u_h - R_h w^H, \Omega) \le \tilde{\eta}_h(R_h w^H, \Omega) + \eta_h(u_h - R_h w^H, \Omega)$$

$$\le \tilde{\eta}_h(R_h w^H, \Omega) + \mathcal{O}(r(h_0))(\|u - u_h\|_{a,\Omega} + \|u - u_H\|_{a,\Omega})$$

leads to the desired result (4.3) and the proof is complete. $\qquad \square$

Now we are at the position to give the error reduction of **Adaptive Algorithm C** for the eigenvalue computations.

Theorem 4.1. *For the successive eigenfunction approximations u_H and u_h produced by **Adaptive Algorithm** C, there exist constants $\gamma > 0$, α_0, and $\alpha \in (0,1)$, depending only on the shape regularity of meshes, C_a, c_a and the parameter θ used by **Adaptive Algorithm** C, such that*

$$\|u - u_h\|_{a,\Omega}^2 + \gamma \eta_h^2(u_h, \Omega) \leq \alpha^2 \big(\|u - u_H\|_{a,\Omega}^2 + \gamma \eta_H^2(u_H, \Omega) \big)$$
$$+ \alpha_0^2 r^2(h_0) \|u - u_{H_{-1}}\|_{a,\Omega}, \tag{4.9}$$

provided $h_0 \ll 1$.

Proof. Since $w^h = \lambda_h K u_h$ and $w^H = \lambda_H K u_H$, we conclude from Theorem 2.1 that there exists constant $\tilde{\gamma} > 0$ and $\xi \in (0,1)$ such that

$$\|w^H - R_h w^H\|_{a,\Omega}^2 + \tilde{\gamma} \tilde{\eta}_h^2(R_h w^H, \Omega) \leq \xi^2 \big(\|w^H - R_H w^H\|_{a,\Omega}^2$$
$$+ \tilde{\gamma} \tilde{\eta}_H^2(R_H w^H, \Omega) \big). \tag{4.10}$$

From (4.1) and (4.3), there exists a constant $\hat{C} > 0$ such that

$$\|u - u_h\|_{a,\Omega}^2 + \tilde{\gamma} \eta_h^2(u_h, \Omega)$$
$$\leq (1 + \delta_1) \|w^H - R_h w^H\|_{a,\Omega}^2 + \hat{C}(1 + \delta_1^{-1}) r^2(h_0)(\|u - u_h\|_{a,\Omega}^2 + \|u - u_H\|_{a,\Omega}^2)$$
$$+ (1 + \delta_1) \tilde{\gamma} \tilde{\eta}_h^2(R_h w^H, \Omega) + \hat{C}\tilde{\gamma}(1 + \delta_1^{-1}) r^2(h_0)(\|u - u_h\|_{a,\Omega}^2 + \|u - u_H\|_{a,\Omega}^2)$$
$$\leq (1 + \delta_1) \big(\|w^H - R_h w^H\|_{a,\Omega}^2 + \tilde{\gamma} \tilde{\eta}_h^2(R_h w^H, \Omega) \big)$$
$$+ C_4 \delta_1^{-1} r^2(h_0)(\|u - u_h\|_{a,\Omega}^2 + \|u - u_H\|_{a,\Omega}^2), \tag{4.11}$$

where C_4 depends on the constants $\hat{C}$ and $\tilde{\gamma}$ and the Young inequality is used with $\delta_1 \in (0,1)$ satisfying

$$(1 + \delta_1)\xi^2 < 1.$$

The similar argument leads to

$$\|w^H - R_H w^H\|_{a,\Omega}^2 + \tilde{\gamma} \tilde{\eta}_h^2(R_H w^H, \Omega)$$
$$\leq (1 + \delta_2) \|u - u_H\|_{a,\Omega}^2 + \hat{C}(1 + \delta_2^{-1}) r^2(h_0)(\|u - u_H\|_{a,\Omega}^2 + \|u - u_{H_{-1}}\|_{a,\Omega}^2)$$
$$+ (1 + \delta_2) \tilde{\gamma} \eta_H^2(u_H, \Omega) + \hat{C}\tilde{\gamma}(1 + \delta_2^{-1}) r^2(h_0)(\|u - u_H\|_{a,\Omega}^2 + \|u - u_{H_{-1}}\|_{a,\Omega}^2)$$
$$= (1 + \delta_2) \big(\|u - u_H\|_{a,\Omega}^2 + \tilde{\gamma} \eta_H^2(u_H, \Omega) \big)$$
$$+ C_4 \delta_2^{-1} r^2(h_0)(\|u - u_H\|_{a,\Omega}^2 + \|u - u_{H_{-1}}\|_{a,\Omega}^2), \tag{4.12}$$

where $u_{H_{-1}}$ denotes the eigenfunction approximation obtained on the mesh level $\mathcal{T}_{h_{k-1}}$ before $\mathcal{T}_H$ and $\delta_2 \in (0,1)$ satisfies

$$(1 + \delta_1)(1 + \delta_2 + C_4 \delta_2^{-1} r^2(h_0))\xi^2 < 1. \tag{4.13}$$

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Combing (4.10) and (4.11) leads to

$$\|u - u_h\|_{a,\Omega}^2 + \tilde{\gamma}\eta_h^2(u_h, \Omega)$$

$$\leq (1 + \delta_1)\xi^2\big(\|w^H - R_H w^H\|_{a,\Omega}^2 + \tilde{\gamma}\tilde{\eta}_H^2(R_H w^H, \Omega)\big)$$

$$+ C_4\delta_1^{-1}r^2(h_0)\big(\|u - u_h\|_{a,\Omega}^2 + \|u - u_H\|_{a,\Omega}^2\big). \tag{4.14}$$

From (4.12) and (4.14), we have

$$\|u - u_h\|_{a,\Omega}^2 + \tilde{\gamma}\eta_h^2(u_h, \Omega) \leq (1 + \delta_1)\xi^2\Big((1 + \delta_2)\big(\|u - u_H\|_{a,\Omega}^2 + \tilde{\gamma}\eta_H^2(u_H, \Omega)\big)$$

$$+ C_4\delta_2^{-1}r^2(h_0)\big(\|u - u_H\|_{a,\Omega}^2 + \|u - u_{H_{-1}}\|_{a,\Omega}^2\big)\Big)$$

$$+ C_4\delta_1^{-1}r^2(h_0)\Big(\|u - u_h\|_{a,\Omega}^2 + \|u - u_H\|_{a,\Omega}^2\Big). \tag{4.15}$$

Consequently,

$$\big(1 - C_4\delta_1^{-1}r^2(h_0)\big)\|u - u_h\|_{a,\Omega}^2 + \tilde{\gamma}\eta_h^2(u_h, \Omega)$$

$$\leq \big((1 + \delta_1)(1 + \delta_2 + C_4\delta_2^{-1}r^2(h_0))\xi^2 + C_4\delta_1^{-1}r^2(h_0)\big)\|u - u_H\|_{a,\Omega}^2$$

$$+ (1 + \delta_1)(1 + \delta_2)\xi^2\tilde{\gamma}\eta_H^2(u_H, \Omega)$$

$$+ C_4(1 + \delta_1)\delta_2^{-1}\xi^2 r^2(h_0)\|u - u_{H_{-1}}\|_{a,\Omega}^2, \tag{4.16}$$

that is

$$\|u - u_h\|_{a,\Omega}^2 + \frac{\tilde{\gamma}}{1 - C_4\delta_1^{-1}r^2(h_0)}\eta_H^2(u_h, \Omega)$$

$$\leq \frac{(1 + \delta_1)(1 + \delta_2 + C_4\delta_2^{-1}r^2(h_0))\xi^2 + C_4\delta_1^{-1}r^2(h_0)}{1 - C_4\delta_1^{-1}r^2(h_0)}\|u - u_H\|_{a,\Omega}^2$$

$$+ \frac{(1 + \delta_1)(1 + \delta_2)\xi^2\tilde{\gamma}}{1 - C_4\delta_1^{-1}r^2(h_0)}\eta_H^2(u_H, \Omega)$$

$$+ \frac{C_4(1 + \delta_1)\delta_2^{-1}\xi^2}{1 - C_4\delta_1^{-1}r^2(h_0)}r^2(h_0)\|u - u_{H_{-1}}\|_{a,\Omega}^2. \tag{4.17}$$

Since $h_0 \ll 1$ implies $r(h_0) \ll 1$, we have that the constant α defined by

$$\alpha := \left(\frac{(1 + \delta_1)(1 + \delta_2 + C_4\delta_2^{-1}r^2(h_0))\xi^2 + C_4\delta_1^{-1}r^2(h_0)}{1 - C_4\delta_1^{-1}r^2(h_0)}\right)^{1/2} \tag{4.18}$$

satisfying $\alpha \in (0, 1)$. Therefore

$$\|u - u_h\|_{a,\Omega}^2 + \frac{\tilde{\gamma}}{1 - C_4\delta_1^{-1}r^2(h_0)}\eta_H^2(u_h, \Omega) \leq \alpha^2\Big(\|u - u_H\|_{a,\Omega}^2$$

$$+ \frac{(1 + \delta_1)(1 + \delta_2)\xi^2\tilde{\gamma}}{(1 + \delta_1)(1 + \delta_2 + C_4\delta_2^{-1}r^2(h_0))\xi^2 + C_4\delta_1^{-1}r^2(h_0)}\eta_H^2(u_H, \Omega)\Big)$$

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$$+ \frac{C_4(1+\delta_1)\delta_2^{-1}\xi^2 r^2(h_0)}{1 - C_4\delta_1^{-1}r^2(h_0)}\|u - u_{H_{-1}}\|_{a,\Omega}^2. \tag{4.19}$$

If we choose

$$\gamma := \frac{\tilde{\gamma}}{1 - C_4\delta_1^{-1}r^2(h_0)}, \tag{4.20}$$

we arrive at (4.9) by using the fact

$$\frac{(1+\delta_1)(1+\delta_2)\xi^2\tilde{\gamma}}{(1+\delta_1)(1+\delta_2+C_4\delta_2^{-1}r^2(h_0))\xi^2 + C_4\delta_1^{-1}r^2(h_0)} < \gamma$$

and setting

$$\alpha_0^2 = \frac{C_4(1+\delta_1)\delta_2^{-1}\xi^2}{1 - C_4\delta_1^{-1}r^2(h_0)}.$$

Hence the proof is complete. $\qquad\square$

Based on Theorem 4.1, we can give the following error estimate for **Adaptive Algorithm C**.

Theorem 4.2. *Suppose $(\lambda, u) \in \mathcal{R} \times H_0^1(\Omega)$ be a simple eigenpair of (3.2), (λ_{h_k}, u_{h_k}) be a sequence of finite element solutions produced by* **Adaptive Algorithm C**. *When h_0 is small enough, there exist constants $\beta > 0$ and $\bar{\alpha} \in (0,1)$, depending on the shape regularity of meshes and the parameter θ, such that for any two consecutive iterates k and $k+1$*

$$d_{h_{k+1}}^2 \le \alpha^2 d_{h_k}^2 + \alpha_0^2 r^2(h_0)\|u - u_{h_{k-1}}\|_{a,\Omega}^2, \tag{4.21}$$

and

$$d_{h_{k+1}}^2 + \beta^2 r^2(h_0)d_{h_k}^2 \le \bar{\alpha}^2\big(d_{h_k}^2 + \beta^2 r^2(h_0)d_{h_{k-1}}^2\big), \tag{4.22}$$

where $d_{h_k}^2 = \|u - u_{h_k}\|_{a,\Omega}^2 + \gamma\eta_{h_k}^2(u_{h_k},\Omega)$. Then, **Adaptive Algorithm C** *converges with a linear rate $\bar{\alpha}$, i.e. the n-th iteration solution (λ_{h_n}, u_{h_n}) of* **Adaptive Algorithm C** *has the following error estimates*

$$d_{h_n}^2 + \beta^2 r^2(h_0)d_{h_{n-1}}^2 \le C_0\bar{\alpha}^{2n}, \tag{4.23}$$

$$|\lambda_{h_n} - \lambda| \lesssim \bar{\alpha}^{2n}, \tag{4.24}$$

where $C_0 = \|u - u_{h_0}\|_{a,\Omega}^2 + \gamma\eta_{h_0}^2(u_{h_0},\Omega)$.

Proof. It is obvious that (4.21) can be derived directly from Theorem 4.1. Now we choose $\bar{\alpha}$ and β such that

$$\bar{\alpha}^2 - \beta^2 r^2(h_0) = \alpha^2,$$

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$$\bar{\alpha}^2 \beta^2 \;=\; \alpha_0^2.$$

This equation leads to

$$\bar{\alpha}^2 = \frac{\alpha^2 + \sqrt{\alpha^4 + 4\alpha_0^2 r^2(h_0)}}{2} \;\; \text{and} \;\; \beta^2 = \frac{2\alpha_0^2}{\alpha^2 + \sqrt{\alpha^4 + 4\alpha_0^2 r^2(h_0)}}.$$

As we know $\bar{\alpha} < 1$ provided that $\alpha < 1$ and h_0 is small enough. Then (4.22) can be obtained with the chosen constants $\bar{\alpha}$ and β.

The estimates (4.23) and (4.24) are natural results from (4.22) and (3.3) and the proof is complete. $\qquad\square$

5 Complexity analysis

Due to Theorems 2.1 and 4.1, we are able to analyze the complexity of **Adaptive Algorithm** C for eigenvalue problem via the complexity result of the associated boundary value problems.

In this section, we assume the initial mesh size h_0 is small enough such that

$$r(h_0)\|u - u_{h_{k-1}}\|_{a,\Omega}^2 \leq \|u - u_{h_k}\|_{a,\Omega}^2. \tag{5.1}$$

Then from Theorem 4.1, we have the following error reduction property of **Adaptive Algorithm** C

$$\|u - u_{h_{k+1}}\|_{a,\Omega}^2 + \gamma \eta_{h_{k+1}}^2(u_{h_{k+1}}, \Omega) \leq \tilde{\alpha}^2\big(\|u - u_{h_k}\|_{a,\Omega}^2 + \gamma \eta_{h_k}^2(u_{h_k}, \Omega)\big) \tag{5.2}$$

with $\tilde{\alpha}^2 = \alpha^2 + \alpha_0^2 r(h_0)$.

Based on this contraction result, we also give the complexity analysis with the similar way of [14] and [20]. Let $\mathcal{T}_{h_k}(k \geq 0)$ be the sequence of conforming nested partitions generated by **REFINE** starting from $\mathcal{T}_{h_0}$ with $h_0 \ll 1$. We denote $\mathcal{T}_{h_{k,*}}$ a refinement of $\mathcal{T}_{h_k}$ (in general nonconforming), $\mathcal{M}(\mathcal{T}_{h_k})$ the set of elements of $\mathcal{T}_{h_k}$ that were refined in $\mathcal{T}_{h_k}$. Let $I_{h_{k+1}} : C(\Omega) \cap H_0^1(\Omega) \rightarrow V_{h_{k+1}}$ satisfy

$$I_{h_{k+1}} v = v \;\; \text{on } T \notin \mathcal{M}(\mathcal{T}_{h_k}) \;\; \forall v \in V_{h_{k+1}}$$

and set

$$V_{h_{k,*}} = \begin{cases} V_{h_k} \cup \big((I - I_{h_{k+1}})V_{h_{k+1}}\big), & \text{if } \mathcal{T}_{h_{k,*}} \text{ is nonconforming,} \\ V_{h_{k+1}}, & \text{if } \mathcal{T}_{h_{k,*}} \text{ is conforming.} \end{cases}$$

In our analysis, we also need the following result (see, e.g., [20, 14, 36, 40, 41]).

Lemma 5.1. *(Complexity of **REFINE**) Let $\mathcal{T}_{h_k}$ $(k \geq 0)$ be a sequence of conforming nested partitions generated by **REFINE** starting from $\mathcal{T}_{h_0}$, $\mathcal{M}(T_{h_{k,*}})$ the set of*

elements of $\mathcal{T}_{h_k}$ which is marked for refinement and $\mathcal{T}_{h_{k,*}}$ be the partition created by refinement of elements only in $\mathcal{M}(T_{h_{k,*}})$. There exists a constant $\hat{C}_0$ depending solely on $\mathcal{T}_{h_0}$ such that

$$\#\mathcal{T}_{h_{k+1}} - \#\mathcal{T}_{h_0} \le \hat{C}_0 \sum_{i=0}^{k} \left(\#T_{h_{i,*}} - \#T_{h_i} \right). \tag{5.3}$$

Here and hereafter in this paper, we use $\#\mathcal{T}$ to denote the number of elements in the mesh $\mathcal{T}$.

In order to analyze the complexity of **Adaptive Algorithm** C, we first review some results related to the analysis of complexity for the boundary value problem (2.5). For the proofs, please read the papers [14] and [20].

Lemma 5.2. ([14]) Let $R_{h_k}u \in V_{h_k}$ and $R_{h_{k,*}}u \in V_{h_{k,*}}$ be the discrete solutions of (2.5) on the meshes $\mathcal{T}_{h_k}$ and its refinement $\mathcal{T}_{h_{k,*}}$ with marked element $\mathcal{M}(\mathcal{T}_{h_{k,*}})$. Then we have

$$\|R_{h_k}u - R_{h_{k,*}}u\|_{a,\Omega}^2 \le \tilde{C}_1 \sum_{T \in \mathcal{M}(\mathcal{T}_{h_{k,*}})} \tilde{\eta}_{h_k}^2(R_{h_k}u, T). \tag{5.4}$$

Lemma 5.3. ([14, 20]) Under the same assumptions as in Lemma 5.2 and the energy decrease property

$$\|u - R_{h_{k,*}}u\|_{a,\Omega}^2 + \tilde{\gamma}_0 \mathrm{osc}(f - L(R_{h_{k,*}}u))^2$$
$$\le \tilde{\xi}_0^2(\|u - R_{h_k}u\|_{a,\Omega}^2 + \tilde{\gamma}_0 \mathrm{osc}(f - L(R_{h_k}u))^2) \tag{5.5}$$

with $\tilde{\gamma}_0 > 0$ and $\tilde{\xi}_0^2 \in (0, \frac{1}{2})$. Then the set $\mathcal{M}(\mathcal{T}_{h_{k,*}})$ of marked elements satisfy the Dörfler property

$$\sum_{T \in \mathcal{M}(\mathcal{T}_{h_{k,*}})} \tilde{\eta}_{h_k}^2(R_{h_k}u, T) \ge \tilde{\theta} \sum_{T \in \mathcal{T}_{h_k}} \tilde{\eta}_{h_k}^2(R_{h_k}u, T), \tag{5.6}$$

where $\tilde{\theta} = \frac{\tilde{C}_2^2(1-2\tilde{\xi}_0^2)}{\tilde{C}_0(\tilde{C}_1^2+(1+2C_L^2\tilde{C}_1^2)\tilde{\gamma}_0)}$ with $\tilde{C}_0 = \max\{1, \frac{\tilde{C}_3^2}{\tilde{\gamma}_0}\}$.

As in the normal analysis of AFEM for boundary value problems, we introduce a function approximation class as follows

$$\mathcal{A}_\gamma^s := \left\{ v \in H_0^1(\Omega) : |v|_{s,\gamma} < \infty \right\},$$

where $\gamma > 0$ is a constant and

$$|v|_{s,\gamma} = \sup_{\varepsilon > 0} \varepsilon \inf_{\{\mathcal{T} \subset \mathcal{T}_{h_0} : \inf(\|v - v_\mathcal{T}\|_1^2 + (\gamma+1)\mathrm{osc}(L(v_\mathcal{T}, \mathcal{T}))^2)^{1/2} \le \varepsilon\}} (\#\mathcal{T} - \#\mathcal{T}_{h_0})^s$$

and $\mathcal{T} \subset \mathcal{T}_{h_0}$ means $\mathcal{T}$ is a refinement of $\mathcal{T}_{h_0}$. From the definition, for $\gamma > 0$, we see that $\mathcal{A}^s_\gamma = \mathcal{A}^s_1$ and we denote $\mathcal{A}^s$ as $\mathcal{A}^s_1$, $|v|_s$ as $|v|_{s,\gamma}$ for simplicity. Hence the symbol $\mathcal{A}^s$ is the class of functions that can be approximated within a given tolerance ε by continuous piecewise polynomial functions over a partition $\mathcal{T}$ with the number of degrees of freedom $\#\mathcal{T} - \#\mathcal{T}_{h_0} \lesssim \varepsilon^{-1/s}|v|_s^{1/s}$.

In order to give the proof of optimal complexity of **Adaptive Algorithm** C for the eigenvalue problem (3.1), we should give some preparations. Associated with the eigenpair approximation (λ_{h_k}, u_{h_k}) of (3.4) in the mesh $\mathcal{T}_{h_k}$, we define $w^{h_k} = K(\lambda_{h_k} u_{h_k})$ as in (3.19).

Using the assumption (5.1) and the similar procedure as in the proof of Theorem 4.1 when (4.3) is replaced by (4.2), we have

Lemma 5.4. *Let $(\lambda_{h_k}, u_{h_k}) \in \mathcal{R} \times V_{h_k}$ and $(\lambda_{h_{k,*}}, u_{h_{k,*}}) \in \mathcal{R} \times V_{h_{k,*}}$ be the discrete solutions of (3.1) produced by **Adaptive Algorithm** C over a conforming mesh $\mathcal{T}_{h_k}$ and its (nonconforming) refinement $\mathcal{T}_{h_{k,*}}$ with marked element $\mathcal{M}(\mathcal{T}_{h_{k,*}})$. Supposing they satisfy the following property*

$$\|u - u_{h_{k,*}}\|^2_{a,\Omega} + \gamma_*\mathrm{osc}(L(u_{h_{k,*}}), \mathcal{T}_{h_{k,*}})^2$$
$$\leq \beta^2_*(\|u - u_{h_k}\|^2_{a,\Omega} + \gamma_*\mathrm{osc}(L(u_{h_k}), \mathcal{T}_{h_k})^2), \tag{5.7}$$

where $\gamma_* > 0$, $\beta_* > 0$ are some constants. Then the associated boundary value problem approximations $R_{h_k}w^{h_k}$ and $R_{h_{k,*}}w^{h_k}$ of w^{h_k} have the following contraction property

$$\|w^{h_k} - R_{h_{k,*}}w^{h_k}\|^2_{a,\Omega} + \gamma_*\mathrm{osc}(L(R_{h_{k,*}}w^{h_k}), \mathcal{T}_{h_{k,*}})^2$$
$$\leq \tilde{\beta}^2_*(\|w^{h_k} - R_{h_k}w^{h_k}\|^2_{a,\Omega} + \gamma_*\mathrm{osc}(L(u_{h_k}), \mathcal{T}_{h_k})^2) \tag{5.8}$$

with

$$\tilde{\beta}_* := \left(\frac{(1 + \delta_1 + C_4\delta_1^{-1}r^2(h_0))\beta^2_* + C_4\delta_1^{-1}r^2(h_0)}{1 - C_4r(h_0)(1 + r(h_0))}\right)^{1/2}, \tag{5.9}$$

where the constant C_4 depends on $\delta_1 \in (0,1)$ and γ_* as in the proof of Theorem 4.1.

Proof. From (2.20), (4.1), and (4.2), there exists a constant $\hat{C} > 0$ such that

$$\|w^{h_k} - R_{h_{k,*}}w^{h_k}\|^2_{a,\Omega} + \gamma_*\mathrm{osc}(L(R_{h_{k,*}}w^{h_k}), \mathcal{T}_{h_{k,*}})^2$$
$$\leq (1 + \delta_1)\|u - u_{h_{k,*}}\|^2_{a,\Omega} + \hat{C}(1 + \delta_1^{-1})r^2(h_0)(\|u - u_{h_{k,*}}\|^2_{a,\Omega} + \|u - u_{h_k}\|^2_{a,\Omega})$$
$$+ (1 + \delta_1)\gamma_*\mathrm{osc}(L(u_{h_{k,*}}), \mathcal{T}_{h_{k,*}})^2 + \hat{C}(1 + \delta_1^{-1})r^2(h_0)(\|u - u_{h_{k,*}}\|^2_{a,\Omega}$$
$$+ \|u - u_{h_k}\|^2_{a,\Omega})$$
$$\leq (1 + \delta_1)(\|u - u_{h_{k,*}}\|^2_{a,\Omega} + \gamma_*\mathrm{osc}(L(u_{h_{k,*}}), \mathcal{T}_{h_k})^2)$$
$$+ C_4\delta_1^{-1}r^2(h_0)(\|u - u_{h_{k,*}}\|^2_{a,\Omega} + \|u - u_{h_k}\|^2_{a,\Omega})$$

$$\leq (1+\delta_1)\beta_*^2\big(\|u-u_{h_k}\|_{a,\Omega}^2 + \gamma_*\text{osc}(L(u_{h_k}),\mathcal{T}_{h_k})^2\big)$$
$$+C_4\delta_1^{-1}r^2(h_0)\beta_*^2\big(\|u-u_{h_k}\|_{a,\Omega}^2 + \gamma_*\text{osc}(L(u_{h_k}),\mathcal{T}_{h_k})^2\big)$$
$$+C_4\delta_1^{-1}r^2(h_0)\|u-u_{h_k}\|_{a,\Omega}^2$$
$$\leq \big((1+\delta_1+C_4\delta_1^{-1}r^2(h_0))\beta_*^2 + C_4\delta_1^{-1}r^2(h_0)\big)\Big(\|u-u_{h_k}\|_{a,\Omega}^2$$
$$+\gamma_*\text{osc}(L(u_{h_k}),\mathcal{T}_{h_k})^2\Big), \qquad (5.10)$$

where C_4 depends on the constants $\hat{C}$ and γ_* and the Young inequality is used.

Similarly from (2.20), (4.1), (4.2), and (5.1), we have

$$\|u-u_{h_k}\|_{a,\Omega}^2 + \gamma_*\text{osc}(L(u_{h_k}),\mathcal{T}_{h_k})^2$$
$$\leq \|w^{h_k}-R_{h_k}w^{h_k}\|_{a,\Omega}^2 + \gamma_*\text{osc}(L(R_{h_k}w^{h_k}),\mathcal{T}_{h_k})^2$$
$$+C_4r^2(h_0)\big(\|u-u_{h_k}\|_{a,\Omega}^2 + \|u-u_{h_{k-1}}\|_{a,\Omega}^2\big)$$
$$\leq \|w^{h_k}-R_{h_k}w^{h_k}\|_{a,\Omega}^2 + \gamma_*\text{osc}(L(R_{h_k}w^{h_k}),\mathcal{T}_{h_k})^2$$
$$+C_4r(h_0)(1+r(h_0))\|u-u_{h_k}\|_{a,\Omega}^2.$$

Then the following inequality holds

$$\|u-u_{h_k}\|_{a,\Omega}^2 + \gamma_*\text{osc}(L(u_{h_k}),\mathcal{T}_{h_k})^2$$
$$\leq \frac{1}{1-C_4r(h_0)(1+r(h_0))}\big(\|w^{h_k}-R_{h_k}w^{h_k}\|_{a,\Omega}^2$$
$$+\gamma_*\text{osc}(L(R_{h_k}w^{h_k}),\mathcal{T}_{h_k})^2\big). \qquad (5.11)$$

Combining (5.10) and (5.11) leads to

$$\|w^{h_k}-R_{h_{k,*}}w^{h_k}\|_{a,\Omega}^2 + \gamma_*\text{osc}(L(R_{h_{k,*}}w^{h_k}))^2$$
$$\leq \frac{(1+\delta_1+C_4\delta_1^{-1}r^2(h_0))\beta_*^2 + C_4\delta_1^{-1}r^2(h_0)}{1-C_4r(h_0)(1+r(h_0))}\big(\|w^{h_k}-R_{h_k}w^{h_k}\|_{a,\Omega}^2$$
$$+\gamma_*\text{osc}(L(R_{h_k}w^{h_k}),\mathcal{T}_{h_k})^2\big). \qquad (5.12)$$

This is the desired result (5.8) and the proof is complete. $\qquad\square$

We present the following statement which is a direct consequence of Lemmas 5.3 and 5.4.

Corollary 5.1. *Let* $(\lambda_{h_k},u_{h_k}) \in \mathcal{R} \times V_{h_k}$ *and* $(\lambda_{h_{k,*}},u_{h_{k,*}}) \in \mathcal{R} \times V_{h_{k,*}}$ *be as in Lemma 5.4. Suppose they satisfy the following decrease property*

$$\|u-u_{h_{k,*}}\|_{a,\Omega}^2 + \gamma_*\text{osc}(L(u_{h_{k,*}}),\mathcal{T}_{h_{k,*}})^2 \leq \beta_*^2\big(\|u-u_{h_k}\|_{a,\Omega}^2 + \gamma_*\text{osc}(L(u_{h_k}),\mathcal{T}_{h_k})^2\big),$$

where the constants $\gamma_* > 0$ *and* $\beta_*^2 \in (0,1/2)$*. Then the set* $\mathcal{M}(\mathcal{T}_{h_{k,*}})$ *of marked elements satisfies the following inequality*

$$\sum_{T \in \mathcal{M}(\mathcal{T}_{h_{k,*}})} \eta_{h_k}^2(u_{h_k},T) \geq \hat{\theta} \sum_{T \in \mathcal{T}_{h_k}} \eta_{h_k}^2(u_{h_k},T), \qquad (5.13)$$

where the constant $\hat{\theta} = \frac{\tilde{C}_2^2(1-2\tilde{\beta}_*^2)}{\tilde{C}_0(\tilde{C}_1^2+(1+2C_L^2\tilde{C}_1^2)\gamma_*)}$ *and* $\tilde{C}_0 = \max\{1, \frac{\tilde{C}_3^2}{\gamma_*}\}$ *with* $\tilde{\beta}_*$ *and* γ_* *which are the same as in (5.8) and (5.9) with* δ_1 *being chosen such that* $\tilde{\beta}_*^2 \in (0, 1/2)$.

Lemma 5.5. *(Upper Bound of DOF). Let* $u \in \mathcal{A}^s$ *and* $\mathcal{T}_{h_k}$ *be a conforming partition from* $\mathcal{T}_{h_0}$. *Let* $\mathcal{T}_{h_{k,*}}$ *be a mesh created from* $\mathcal{T}_{h_k}$ *by marking the set* $\mathcal{M}(\mathcal{T}_{h_{k,*}})$ *according to* **Marking Strategy** E *with* $\theta \in (0, \frac{C_2^2\gamma}{C_3^2(C_1^2+(1+2C_L^2C_1^2)\gamma)})$. *Then we have*

$$\#\mathcal{T}_{h_{k,*}} - \#\mathcal{T}_{h_0} \leq C\big(\|u - u_{h_k}\|_{a,\Omega}^2 + \gamma\mathrm{osc}(L(u_{h_n}), \mathcal{T}_{h_n})^2\big)^{-1/(2s)}|u|_s^{1/s}, \qquad (5.14)$$

where the constant C *depends on the discrepancy between* θ *and* $\frac{C_2^2\gamma}{C_3^2(C_1^2+(1+2C_L^2C_1^2)\gamma)}$.

Proof. We choose $\beta, \beta_1 \in (0, 1)$ such that $\beta_1 \in (0, \beta)$ and

$$\theta < \frac{C_2^2\gamma}{C_3^2\big(C_1^2 + (1 + 2C_L^2C_1^2)\gamma\big)}(1 - \beta^2).$$

Let

$$\varepsilon = \frac{1}{\sqrt{2}}\beta_1\big(\|u - u_{h_k}\|_{a,\Omega}^2 + \gamma\mathrm{osc}(L(u_{h_k}), \mathcal{T}_{h_k})^2\big)^{1/2}.$$

Let $\delta_1 \in (0, 1)$ and $\delta_2 \in (0, 1)$ be the constants such that (4.13) and

$$(1 + \delta_1)^2(1 + \delta_2)^2\beta_1^2 \leq \beta^2, \qquad (5.15)$$

which implies

$$(1 + \delta_1)(1 + \delta_2)\beta_1^2 < 1. \qquad (5.16)$$

Let $\mathcal{T}_{h_\varepsilon}$ be a refinement of $\mathcal{T}_{h_0}$ with minimum degrees of freedom satisfying

$$\|u - u_{h_\varepsilon}\|_{a,\Omega}^2 + (\gamma + 1)\mathrm{osc}(L(u_{h_\varepsilon}), \mathcal{T}_{h_\varepsilon})^2 \leq \varepsilon^2, \qquad (5.17)$$

where u_{h_ε} denotes the solution of eigenvalue problem (3.4) over the mesh $\mathcal{T}_{h_\varepsilon}$. By the definition of $\mathcal{A}^s$, we can get that

$$\#\mathcal{T}_{h_\varepsilon} - \#\mathcal{T}_{h_0} \leq \left(\frac{1}{\sqrt{2}}\beta_1\right)^{-1/s}\big(\|u - u_{h_k}\|_{a,\Omega}^2 + \gamma\mathrm{osc}(L(u_{h_k}), \mathcal{T}_{h_k})^2\big)^{-1/(2s)}|u|_s^{1/s}. \quad (5.18)$$

Let $\mathcal{T}_{h_{k,+}}$ be the smallest (nonconforming) common refinement of $\mathcal{T}_{h_k}$ and $\mathcal{T}_{h_\varepsilon}$. Since both $\mathcal{T}_{h_k}$ and $\mathcal{T}_{h_\varepsilon}$ are refinements of $\mathcal{T}_{h_0}$, the number of elements in $\mathcal{T}_{h_{k,+}}$ that are not in $\mathcal{T}_{h_k}$ is less than the number of elements that must be added to go from $\mathcal{T}_{h_0}$ to $\mathcal{T}_{h_\varepsilon}$, namely,

$$\#\mathcal{T}_{h_{k,+}} - \#\mathcal{T}_{h_k} \leq \#\mathcal{T}_{h_\varepsilon} - \#\mathcal{T}_{h_0}. \qquad (5.19)$$

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Let $w^{h_\varepsilon} = K(\lambda_{h_\varepsilon} u_{h_\varepsilon})$. From definition, we can easily get

$$
\begin{aligned}
\mathrm{osc}(L(R_{h_{k,+}} w^{h_\varepsilon}), \mathcal{T}_{h_{k,+}}) &\leq \mathrm{osc}(L(R_{h_\varepsilon} w^{h_\varepsilon}), \mathcal{T}_{h_{k,+}}) \\
&\quad + \mathrm{osc}(L(R_{h_{k,+}} w^{h_\varepsilon} - R_{h_\varepsilon} w^{h_\varepsilon}), \mathcal{T}_{h_{k,+}}) \\
&\leq \mathrm{osc}(L(R_{h_\varepsilon} w^{h_\varepsilon}), \mathcal{T}_{h_{k,+}}) \\
&\quad + C_L \| R_{h_{k,+}} w^{h_\varepsilon} - R_{h_\varepsilon} w^{h_\varepsilon} \|_{a,\Omega},
\end{aligned}
\tag{5.20}
$$

where Lemma 2.3 is used. Then by the Young inequality, we have

$$
\begin{aligned}
\mathrm{osc}(L(R_{h_{k,+}} w^{h_\varepsilon}), \mathcal{T}_{h_{k,+}})^2 &\leq 2\mathrm{osc}(L(R_{h_\varepsilon} w^{h_\varepsilon}), \mathcal{T}_{h_{k,+}})^2 \\
&\quad + 2C_L^2 \| R_{h_{k,+}} w^{h_\varepsilon} - R_{h_\varepsilon} w^{h_\varepsilon} \|_{a,\Omega}^2.
\end{aligned}
\tag{5.21}
$$

Since $\mathcal{T}_{h_{k,+}}$ is a refinement of $\mathcal{T}_{h_\varepsilon}$, L^2-projection error are monotone and the following orthogonality

$$
\| w^{h_\varepsilon} - R_{h_{k,+}} w^{h_\varepsilon} \|_{a,\Omega}^2 = \| w^{h_\varepsilon} - R_{h_\varepsilon} w^{h_\varepsilon} \|_{a,\Omega}^2 - \| R_{h_{k,+}} w^{h_\varepsilon} - R_{h_\varepsilon} w^{h_\varepsilon} \|_{a,\Omega}^2
\tag{5.22}
$$

is valid, we arrive at

$$
\begin{aligned}
&\| w^{h_\varepsilon} - R_{h_{k,+}} w^{h_\varepsilon} \|_{a,\Omega}^2 + \frac{1}{2C_L^2} \mathrm{osc}(L(R_{h_{k,+}} w^{h_\varepsilon}), \mathcal{T}_{h_{k,+}})^2 \\
&\leq \| w^{h_\varepsilon} - R_{h_\varepsilon} w^{h_\varepsilon} \|_{a,\Omega}^2 + \frac{1}{C_L^2} \mathrm{osc}(L(R_{h_\varepsilon} w^{h_\varepsilon}), \mathcal{T}_{h_\varepsilon})^2.
\end{aligned}
\tag{5.23}
$$

Note that (2.22) implies $\tilde{\gamma} \leq \frac{1}{2C_L^2}$ and we obtain that

$$
\begin{aligned}
&\| w^{h_\varepsilon} - R_{h_{k,+}} w^{h_\varepsilon} \|_{a,\Omega}^2 + \tilde{\gamma}\,\mathrm{osc}(L(R_{h_{k,+}} w^{h_\varepsilon}), \mathcal{T}_{h_{k,+}})^2 \\
&\leq \| w^{h_\varepsilon} - R_{h_\varepsilon} w^{h_\varepsilon} \|_{a,\Omega}^2 + \frac{1}{C_L^2} \mathrm{osc}(L(R_{h_\varepsilon} w^{h_\varepsilon}), \mathcal{T}_{h_\varepsilon})^2 \\
&\leq \| w^{h_\varepsilon} - R_{h_\varepsilon} w^{h_\varepsilon} \|_{a,\Omega}^2 + (\tilde{\gamma} + \sigma)\,\mathrm{osc}(L(R_{h_\varepsilon} w^{h_\varepsilon}), \mathcal{T}_{h_\varepsilon})^2
\end{aligned}
\tag{5.24}
$$

with $\sigma = \frac{1}{C_L^2} - \tilde{\gamma} \in (0,1)$. Applying the similar argument in the proof of Theorem 4.1 when (4.3) is replaced by (4.2), we then obtain

$$
\begin{aligned}
&\| u - u_{h_{k,+}} \|_{a,\Omega}^2 + \gamma\,\mathrm{osc}(L(u_{h_{k,+}}), \mathcal{T}_{h_{k,+}}) \\
&\leq \beta_0^2 \Big(\| u - u_{h_\varepsilon} \|_{a,\Omega}^2 + (\gamma + \sigma)\,\mathrm{osc}(L(u_{h_\varepsilon}), \mathcal{T}_{h_\varepsilon})^2 \Big) \\
&\leq \beta_0^2 \Big(\| u - u_{h_\varepsilon} \|_{a,\Omega}^2 + (\gamma + 1)\,\mathrm{osc}(L(u_{h_\varepsilon}), \mathcal{T}_{h_\varepsilon})^2 \Big),
\end{aligned}
\tag{5.25}
$$

where

$$
\beta_0 = \left(\frac{(1 + \delta_1)\big((1 + \delta_2) + C_5 \delta_2^{-1} r^2(h_0)\big) + C_5 \delta_1^{-1} r^2(h_0)}{1 - C_5 \delta_1^{-1} r^2(h_0)} \right)^{1/2}
$$

and

$$\gamma = \frac{\tilde{\gamma}}{1 - C_5 \delta_1^{-1} r^2(h_0)}.$$

with C_5 the constant depending on C_L similar to C_4 in the proof of Theorem 4.1. Combing (5.17) and (5.25) leads to

$$\|u - u_{h_{k,+}}\|_{a,\Omega}^2 + \gamma \mathrm{osc}(L(u_{h_{k,+}}), \mathcal{T}_{h_{k,+}}) \leq \check{\beta}\Big(\|u - u_{h_k}\|_{a,\Omega}^2 + \gamma \mathrm{osc}(L(u_{h_k}), \mathcal{T}_{h_k})^2\Big)$$

with $\check{\beta} = \frac{1}{\sqrt{2}}\beta_0\beta_1$.

It is seen from $h_0 \ll 1$ and (5.16) that $\check{\beta}^2 \in (0, \frac{1}{2})$. Thus by Corollary 5.1 we have that $\mathcal{T}_{h_{k,+}}$ satisfies

$$\sum_{T \in \mathcal{M}(\mathcal{T}_{h_{k,+}})} \eta_{h_k}(T)^2 \geq \check{\theta} \sum_{T \in \mathcal{T}_{h_k}} \eta_{h_k}(T)^2,$$

where $\check{\theta} = \frac{\tilde{C}_2^2(1-2\hat{\beta}^2)}{\tilde{C}_0(\tilde{C}_1^2 + (1+2C_L^2\tilde{C}_1^2)\hat{\gamma})}$, $\hat{\gamma} = \frac{\gamma}{1-C_5\delta_1^{-1}r^2(h_0)}$, $\tilde{C}_0 = \max\{1, \frac{\tilde{C}_3^2}{\hat{\gamma}}\}$ and

$$\hat{\beta} = \left(\frac{(1 + \delta_1 + C_5\delta_1^{-1}r^2(h_0))\check{\beta}^2 + C_5\delta_1^{-1}r^2(h_0)}{1 - C_5 r(h_0)(1 + r(h_0))}\right)^{1/2}.$$

From the definition of γ (see (4.20)), we obtain that $\hat{\gamma} < 1$. On the other hand we have $\tilde{C}_3 > 1$ and hence $\tilde{C}_0 = \frac{\tilde{C}_3^2}{\hat{\gamma}}$. Consequently, we can write $\check{\theta}$ as $\check{\theta} = \frac{\tilde{C}_2^2(1-2\hat{\beta}^2)}{\tilde{C}_3^2(\frac{\tilde{C}_1^2}{\hat{\gamma}} + (1+2C_L^2\tilde{C}_1^2))}$.

Since $h_0 \ll 1$ and (5.15), we obtain that $\hat{\gamma} > \gamma$ and $\hat{\beta} \in (0, \frac{1}{\sqrt{2}}\beta)$. Using (3.41) and (3.43), we get

$$
\begin{aligned}
\check{\theta} &= \frac{\tilde{C}_2^2(1-2\hat{\beta}^2)}{\tilde{C}_3^2(\frac{\tilde{C}_1^2}{\hat{\gamma}} + (1+2C_L^2\tilde{C}_1^2))} \geq \frac{\tilde{C}_2^2}{\tilde{C}_3^2\Big(\frac{\tilde{C}_1^2}{\hat{\gamma}} + (1+2C_L^2\tilde{C}_1^2)\Big)}(1 - \beta^2) \\
&= \frac{(2 + \tilde{C}r^2(h_0))C_2^2}{(2 + \tilde{C}r^2(h_0))C_3^2\Big(\frac{\tilde{C}_1^2}{\hat{\gamma}} + (1+2C_L^2\tilde{C}_1^2)\Big)}(1 - \beta^2) \\
&\geq \frac{C_2^2}{C_3^2\Big(\frac{C_1^2}{\gamma} + (1+2C_L^2C_1^2)\Big)}(1 - \beta^2) \\
&= \frac{C_2^2\gamma}{C_3^2\Big(C_1^2 + (1+2C_L^2C_1^2)\gamma\Big)}(1 - \beta^2) > \theta. \quad (5.26)
\end{aligned}
$$

Note that **Marking Strategy** E selects a minimum set $\mathcal{M}(\mathcal{T}_{h_{k,*}})$ satisfying

$$\sum_{T \in \mathcal{M}(\mathcal{T}_{h_{k,*}})} \eta_{h_k}(T)^2 \geq \theta \sum_{T \in \mathcal{T}_{h_k}} \eta_{h_k}(T)^2,$$

which implies that the nonconforming partition $\mathcal{T}_{h_{k,*}}$ satisfies

$$
\begin{aligned}
\#\mathcal{T}_{h_{k,*}} - \#\mathcal{T}_{h_k} &\leq \#\mathcal{T}_{h_{k,+}} - \#\mathcal{T}_{h_k} \leq \#\mathcal{T}_{h_\varepsilon} - \#\mathcal{T}_{h_0} \\
&\leq \left(\frac{1}{\sqrt{2}}\beta_1\right)^{-1/s} \left(\|u - u_{h_k}\|_{a,\Omega}^2 + \gamma\mathrm{osc}(L(u_{h_k}), \mathcal{T}_{h_k})^2\right)^{-1/(2s)} |u|_s^{1/s}. \quad (5.27)
\end{aligned}
$$

This is the desired estimate (5.14) with an explicit dependence on the discrepancy between θ and $\frac{C_2^2\gamma}{C_3^2(C_1^2+(1+2C_L^2C_1^2)\gamma)}$ via β_1. $\qquad\square$

We are now in the position to prove the optimal complexity of **Adaptive Algorithm** C which is stated in the following theorem. Please refer the papers [14] and [20] for the proof.

Theorem 5.1. ([14, 20]) *Let* $(\lambda, u) \in \mathcal{R} \times (H_0^1(\Omega) \cap \mathcal{A}^s)$ *be some simple eigenpair of (3.1) and* $\{(\lambda_{h_k}, u_{h_k})\}_{k\in\mathbb{N}_0}$ *be the sequence of finite element approximations corresponding to the sequence of pairs* $\{\mathcal{R} \times V_{h_k}\}_{k\in\mathbb{N}_0}$ *produced by* **Adaptive Algorithm** C *satisfying (3.10). Then under the assumption (5.1), the n-th iterate solution* (λ_{h_n}, u_{h_n}) *of* **Adaptive Algorithm** C *satisfies the optimal bounds*

$$
\|u - u_{h_k}\|_{a,\Omega}^2 + \gamma\mathrm{osc}(L(u_{h_n}), \mathcal{T}_{h_n})^2 \lesssim (\#\mathcal{T}_{h_n} - \#\mathcal{T}_{h_0})^{-2s}, \quad (5.28)
$$

$$
|\lambda_{h_n} - \lambda| \lesssim (\#\mathcal{T}_{h_n} - \#\mathcal{T}_{h_0})^{-2s}, \quad (5.29)
$$

where the hidden constant depends on the exact eigenpair (λ, u) *and the discrepancy between* θ *and* $\frac{C_2^2\gamma}{C_3^2(C_1^2+(1+2C_L^2C_1^2)\gamma)}$.

Proof. We give the proof in the same way in [20]. From (3.37) and (5.2), we have

$$
\|u - u_{h_k}\|_{a,\Omega}^2 + \gamma\eta_{h_k}^2(u_{h_k}, \Omega) \leq \check{C}\left(\|u - u_{h_k}\|_{a,\Omega}^2 + \gamma\mathrm{osc}(L(u_{h_k}), \mathcal{T}_{h_k})^2\right), \quad (5.30)
$$

where $\check{C}$ depends on C_2, C_3, γ, and $r(h_0)$.

Combining (5.14) and (5.30) leads to

$$
\#\mathcal{T}_{h_{k,*}} - \#\mathcal{T}_{h_k} \leq C\check{C}^{\frac{1}{2s}}\left(\|u - u_{h_k}\|_{a,\Omega}^2 + \gamma\eta_{h_k}(u_{h_k}, \Omega)\right)^{-1/(2s)} |u|^{1/s}. \quad (5.31)
$$

From (5.2) we have

$$
\begin{aligned}
&\left(\|u - u_{h_k}\|_{a,\Omega}^2 + \gamma\eta_{h_k}^2(u_{h_k}, \Omega)\right)^{-1/(2s)} \\
&\qquad\leq \alpha^{(n-k)/s}\left(\|u - u_{h_n}\|_{a,\Omega}^2 + \gamma\eta_{h_n}^2(u_{h_n}, \Omega)\right)^{-1/(2s)}. \quad (5.32)
\end{aligned}
$$

Employing Lemma 5.1, (5.14), and (5.32), we can obtain

$$
\#\mathcal{T}_{h_n} - \#\mathcal{T}_{h_0} \lesssim \sum_{k=0}^{n-1}\left(\#\mathcal{T}_{h_{k,*}} - \mathcal{T}_{h_k}\right)
$$

$$\lesssim |u|_s^{1/s} \sum_{k=0}^{n-1} \left(\|u - u_{h_k}\|_{a,\Omega}^2 + \gamma \eta_{h_k}^2(u_{h_k}, \Omega) \right)^{-1/(2s)}$$

$$\lesssim \left(\|u - u_{h_n}\|_{a,\Omega}^2 + \gamma \eta_{h_n}^2(u_{h_n}, \Omega) \right)^{-1/(2s)} |u|_s^{1/s} \sum_{k=1}^{n} \alpha^{\frac{k}{s}}.$$

Combining the fact $\alpha < 1$ leads to

$$\#\mathcal{T}_{h_n} - \#\mathcal{T}_{h_0} \lesssim \left(\|u - u_{h_n}\|_{a,\Omega}^2 + \gamma \eta_{h_n}^2(u_{h_n}, \Omega) \right)^{-1/(2s)} |u|_s^{1/s}.$$

Sice $\mathrm{osc}(L(u_h), \mathcal{T}_{h_n}) \le \eta_{h_n}(u_{h_n}, \Omega)$, we have

$$\#\mathcal{T}_{h_n} - \#\mathcal{T}_{h_0} \lesssim \left(\|u - u_{h_n}\|_{a,\Omega}^2 + \gamma \mathrm{osc}(u_{h_n}, \mathcal{T}_{h_n})^2 \right)^{-1/(2s)} |u|_s^{1/s}.$$

This is the desired result (5.28) and (5.29) can be derived from (5.28) and Lemma 3.1. Then the proof is complete. $\qquad\square$

6 Numerical experiments

In this section, we present sone numerical examples of **Adaptive Algorithm** C for the second order elliptic eigenvalue problems by the linear finite element method.

Example 1. In this example, we consider the following eigenvalue problem (see [26])

$$\begin{cases} -\frac{1}{2}\Delta u + \frac{1}{2}|x|^2 u &= \lambda u \quad \text{in } \Omega, \\ u &= 0 \quad \text{on } \partial\Omega, \\ \|u\|_{0,\Omega} &= 1, \end{cases} \tag{6.1}$$

where $\Omega = \mathcal{R}^2$ and $|x| = \sqrt{x_1^2 + x_2^2}$. The first eigenvalue of (6.1) is $\lambda = 1$ and the associated eigenfunction is $u = \kappa e^{-|x|^2/2}$ with any nonzero constant κ. In our computation, we set $\Omega = (-5, 5) \times (-5, 5)$.

First, we investigate the numerical results for the first eigenvalue approximations. We give the numerical results for the eigenpair approximation by **Adaptive Algorithm** C with the parameter $\theta = 0.4$. Figure 1 shows the initial triangulation and the triangulation after 14 adaptive iterations. Figure 2 gives the corresponding numerical results for the first 19 adaptive iterations. In order to show the efficiency of **Adaptive Algorithm** C more clearly, we compare the results with those obtained with direct AFEM. It is observed from Figures 2, the approximations of eigenvalue as well as eigenfunction approximations have the optimal convergence rate which coincides with our theory.

Example 2. In the second example, we consider the Laplace eigenvalue problem on the L-shape domain

$$\begin{cases} -\Delta u &= \lambda u \quad \text{in } \Omega, \\ u &= 0 \quad \text{on } \partial\Omega, \\ \|u\|_{0,\Omega} &= 1, \end{cases} \tag{6.2}$$
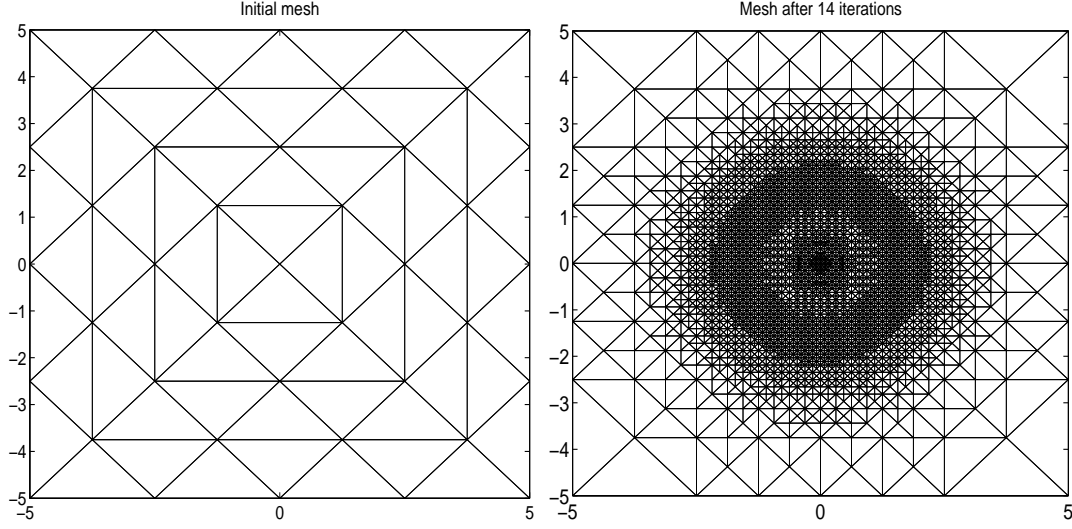
Figure 1: The initial triangulation and the one after 14 adaptive iterations for Example 1

where $\Omega = (-1,1) \times (-1,1) \backslash [0,1) \times (-1,0]$. Since Ω has a reentrant corner, eigenfunctions with singularities are expected. The convergence order for eigenvalue approximations is less than 2 by the linear finite element method which is the order predicted by the theory for regular eigenfunctions.

First, we investigate the numerical results for the first eigenvalue approximations. Since the exact eigenvalue is not known, we choose an adequately accurate approximation $\lambda = 9.6397238440219$ as the exact first eigenvalue for our numerical tests. We give the numerical results for the first eigenpair approximation of **Adaptive Algorithm** C with the parameter $\theta = 0.4$. Figure 3 shows the initial triangulation and the triangulation after 12 adaptive iterations. Figure 4 gives the corresponding numerical results for the first 20 adaptive iterations. In order to show the efficiency of **Adaptive Algorithm** C more clearly, we compare the results with those obtained by direct AFEM. We also test **Adaptive Algorithm** C for 5 smallest eigenvalue approximations and their associated eigenfunction approximations. Figure 5 shows the corresponding a posteriori error estimator $\eta_h(u_h, \Omega)$ produce by **Adaptive Algorithm** C and direct AFEM.

From Figures 4 and 5, we can find the approximations of eigenvalues as well as eigenfunctions have optimal convergence rate which coincides with our theory.

Example 3. In this example, we consider the following second order elliptic eigenvalue problem

$$\begin{cases} -\nabla \cdot (\mathcal{A}\nabla u) + \varphi u & = & \lambda u & \text{in } \Omega, \\ u & = & 0 & \text{on } \partial\Omega, \\ \|u\|_{0,\Omega} & = & 1, \end{cases} \qquad (6.3)$$
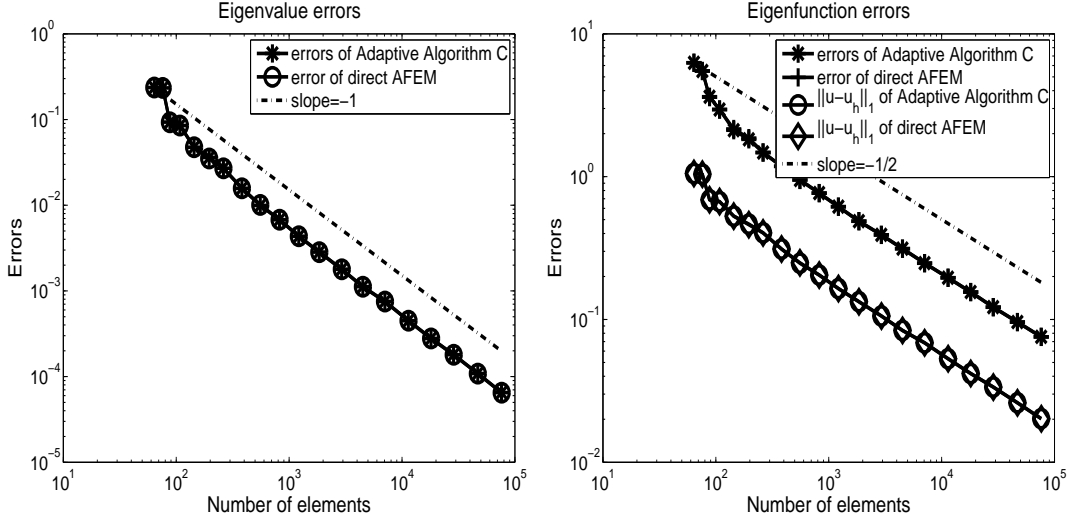
28

Figure 2: The errors of the eigenvalue and the associated eigenfunction approximations by **Adaptive Algorithm** C and direct AFEM for Example 1

with

$$\mathcal{A} = \left(\begin{array}{cc} 1 + (x_1 - \frac{1}{2})^2 & (x_1 - \frac{1}{2})(x_2 - \frac{1}{2}) \\ (x_1 - \frac{1}{2})(x_2 - \frac{1}{2}) & 1 + (x_2 - \frac{1}{2})^2 \end{array} \right),$$

$\varphi = e^{(x_1 - \frac{1}{2})(x_2 - \frac{1}{2})}$ and $\Omega = (-1, 1) \times (-1, 1) \backslash [0, 1) \times (-1, 0]$.

We first investigate the numerical results for the first eigenvalue approximations. Since the exact eigenvalue is not known neither, we choose an adequately accurate approximation $\lambda = 13.58258211870407$ as the exact eigenvalue for our numerical tests. We give the numerical results for the first eigenpair approximation by **Adaptive Algorithm** C with the parameter $\theta = 0.4$. Figure 6 shows the initial triangulation and the triangulation after 12 adaptive iterations. Figure 7 gives the corresponding numerical results for the first 18 adaptive iterations. In order to show the efficiency of **Adaptive Algorithm** C more clearly, we compare the results with those obtained with direct AFEM.

We also test **Adaptive Algorithm** C for 5 smallest eigenvalue approximations and their associated eigenfunction approximations. Figure 8 shows the a posteriori error estimator $\eta_h(u_h, \Omega)$ produced by **Adaptive Algorithm** C and direct AFEM. From Figures 7 and 8, we can find the approximations of eigenvalues as well as eigenfunctions have optimal convergence rate.

7   Concluding remarks

In this paper, we present a type of AFEM for eigenvalue problem based on multilevel correction scheme. The convergence and optimal complexity have also been proved
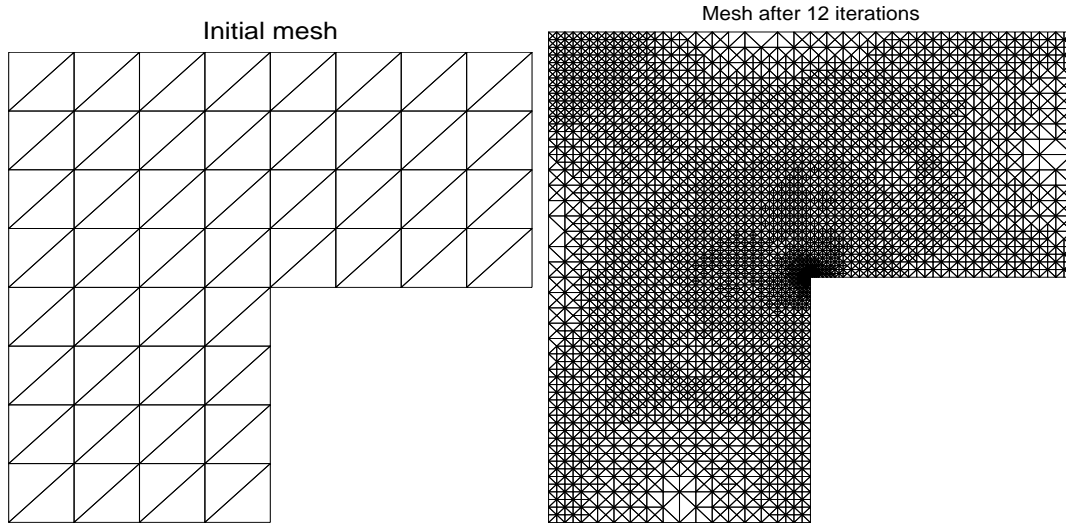
Figure 3: The initial triangulation and the one after 12 adaptive iterations for Example 2

based on a relationship between the eigenvalue problem and the associated boundary value problem (see Theorem 3.1). We also provide some numerical experiments to demonstrate the efficiency of the AFEM for eigenvalue problems.

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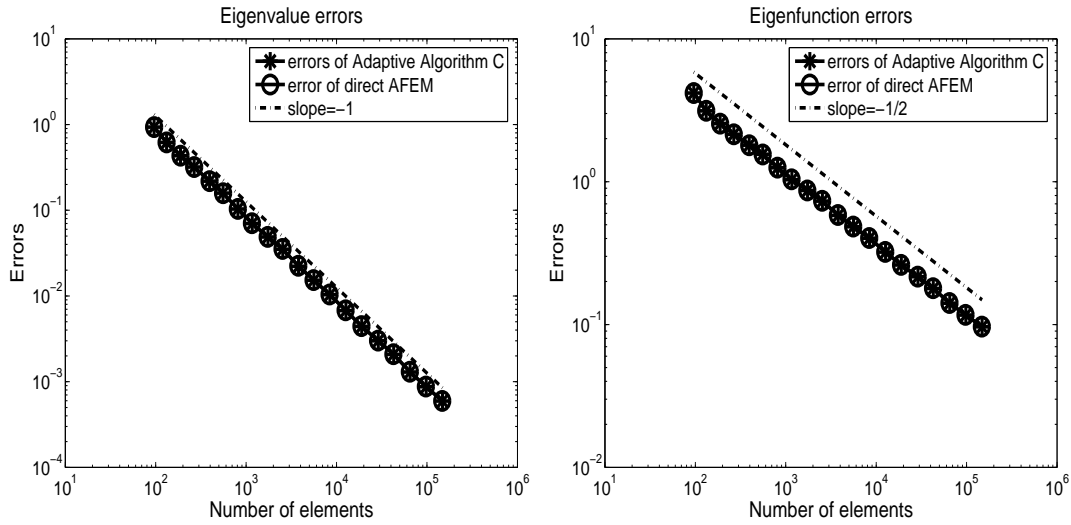
Figure 4: The errors of the smallest eigenvalue approximations and the a posteriori errors of the associated eigenfunction approximations by **Adaptive Algorithm** C and direct AFEM for Example 2

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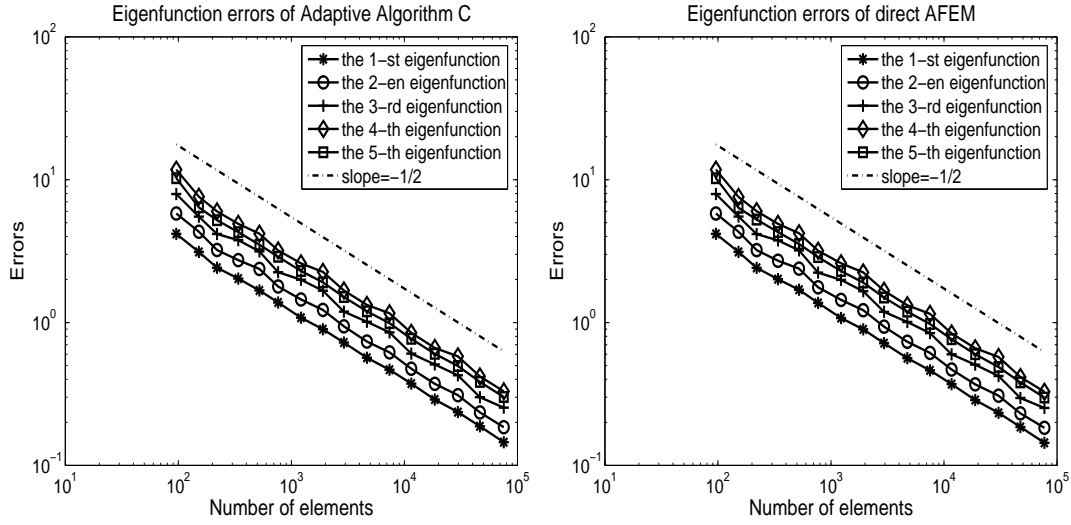[15] Chatelin, F.: Spectral Approximations of Linear Operators. Academic Press, New York (1983)

Figure 5: The a posteriori error estimates of the eigenfunction approximations by **Adaptive Algorithm** C and direct AFEM for Example 2

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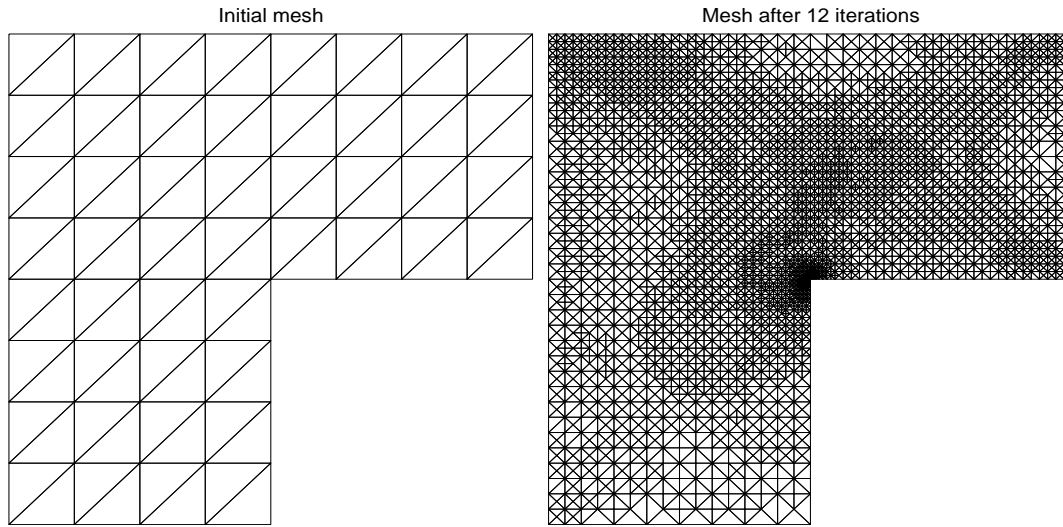
Figure 6: The initial triangulation and the one after 12 adaptive iterations for Example 3

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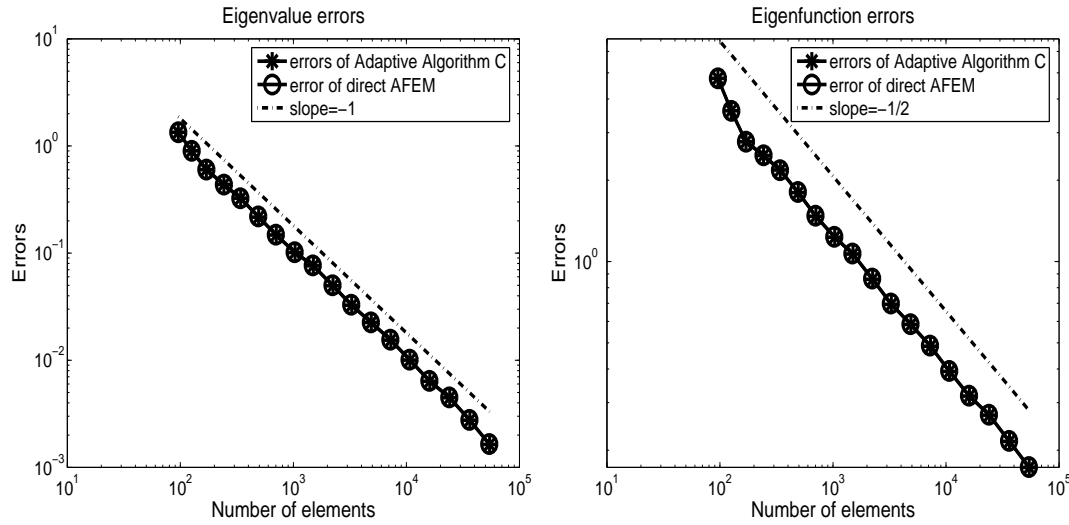
Figure 7: The errors of the smallest eigenvalue and the associated eigenfunction approximations by **Adaptive Algorithm** C and direct AFEM for Example 3

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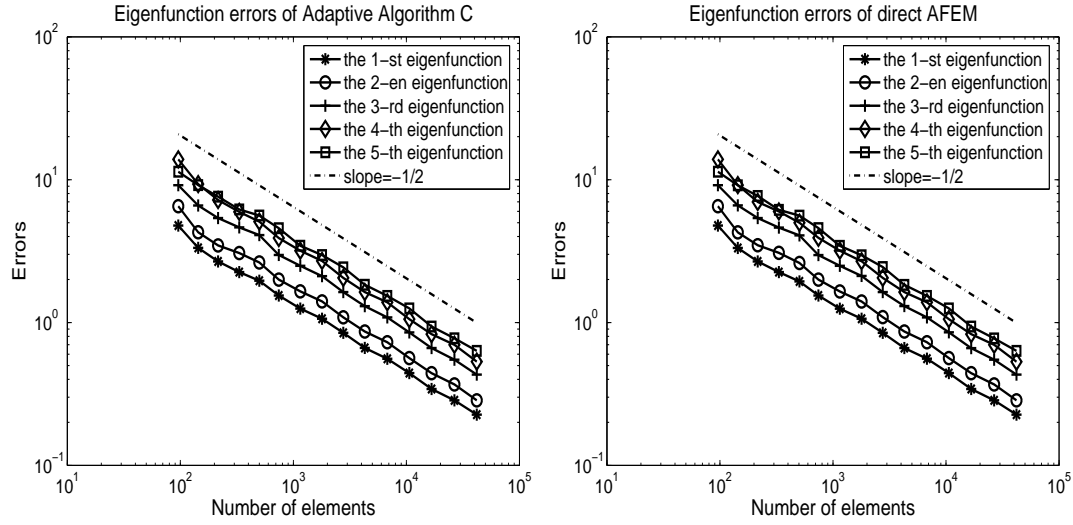Figure 8: The a posteriori error estimates of the eigenfunction approximations by **Adaptive Algorithm** C and direct AFEM for Example 3

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